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ENTROPIC ISOPERIMETRIC INEQUALITIES FOR GENERALIZED FISHER INFORMATION

SERGEY G. BOBKOV AND CYRIL ROBERTO

Abstract. Pursuing an earlier paper on the entropic isoperimetric inequalities, we discuss optimal bounds on the Rényi entropies in terms of the Fisher information of order s.

1. Introduction

Given a random vector X in $\mathbb{R}^n$ with a smooth density p, the Fisher information of order $s \geq 1$ is defined by

$$I_s(X) = \int \Big(\frac{|\nabla p|}{p}\Big)^s\, p,$$

where the integral may be restricted to the supporting set $\mathrm{supp}(p) = \{x : p(x) > 0\}$. Here and elsewhere the integration is understood with respect to Lebesgue measure on $\mathbb{R}^n$. Generalizing the usual Fisher information $I_2 = I$, the functional I_s has been introduced by Vajda [27]. Afterwards it has become a subject of various investigations from the points of view of information and probability theories, statistical estimation and Sobolev-type inequalities, cf. e.g. [8], [20], [3], [10]. In dimension $n = 1$, one may write

$$I_s(X) = \mathbb{E}\,|\rho(X)|^s,$$

where $\rho = (\log p)'$. In the theory of differentiable measures, it is commonly called and treated as a logarithmic derivative of p, cf. [4], [19]. Another name for ρ is the score function, so that $I_s(X)$ may be viewed as the s-th absolute moment of the score of X (cf. [17], [26], [5]).

It should be clear that the map $s \mapsto I_s(X)^{1/s}$ is non-decreasing. Its minimal value

$$I_1(X) = \int |\nabla p|$$

describes the total variation norm of the density p, while $\lim_{s\to\infty} I_s(X)^{1/s} = \sup_{x\in \mathrm{supp}(p)} \frac{|\nabla p(x)|}{p(x)}$ represents the Lipschitz semi-norm of $\log p$.

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The aim of this note is to relate the generalized Fisher information to the Rényi entropy power

$$N_\alpha(X) = \Big(\int p^\alpha\Big)^{-\frac{2}{n(\alpha-1)}} \tag{1.1}$$

of a given order $\alpha \in (0,\infty)$, which is well-defined by this formula whenever $\alpha \neq 1$. Since

$$N_\alpha(X)^{-\frac{n}{2}} = \|p\|_{L^{\alpha-1}(p\,dx)},$$

the map $\alpha \mapsto N_\alpha$ is non-increasing in α. Therefore, by monotonicity, one defines the limit entropy powers

$$\begin{aligned} N_\infty(X) &= \lim_{\alpha\to\infty} N_\alpha(X) \ = \ \|p\|_\infty^{-\frac{2}{n}}, \\ N_0(X) &= \lim_{\alpha\to 0} N_\alpha(X) \ = \ \mathrm{vol}_n(\mathrm{supp}(p))^{\frac{2}{n}}, \end{aligned} \tag{1.2}$$

where $\|p\|_\infty = \mathrm{ess\,sup}\, p(x)$ and vol_n stands for the n-dimensional volume. As a standard approach, one may also put $N_1(X) = \lim_{\alpha\downarrow 1} N_\alpha(X)$ which returns us to the usual definition of the Shannon entropy power

$$N_1(X) = N(X) = \exp\Big\{-\frac{2}{n}\int p\log p\Big\}$$

under mild moment assumptions (such as $N_\alpha(X) > 0$ for some $\alpha > 1$).

Basic fundamental relations we are interested in are inequalities of the form

$$N_\alpha(X)\, I_s(X)^{\frac{2}{s}} \geq c_{\alpha,s,n} \tag{1.3}$$

with (positive) constants that do not depend on p. To stress their universal character, one should note that the above product is invariant under all linear orthogonal transformations of the space, as well as under all affine transformations $X \to Y = a + \lambda X$ ($a \in \mathbb{R}^n$, $\lambda \neq 0$), since then $N_\alpha(Y) = \lambda^2 N_\alpha(X)$ and $I_s(Y) = |\lambda|^{-s} I_s(X)$. In view of the monotonicity properties with respect to α and s mentioned above, such inequalities are getting stronger for the growing parameter α and the decaying parameter s (up to the constants on the right-hand side).

The particular case $\alpha = 1$, $s = 2$ in (1.3) corresponds to the seminal result

$$N(X)\, I(X) \geq 2\pi e\, n \tag{1.4}$$

due to Stam [24], in which the standard normal distribution on $\mathbb{R}^n$ achieves an equality. It may be obtained as a consequence of the entropy power inequality

$$N(X+Y) \geq N(X) + N(Y)$$

by taking $Y = \varepsilon Z$ with Z a standard normal random vector in $\mathbb{R}^n$ independent of X and letting $\varepsilon \to 0$. As was noticed by Costa and Cover [11], in a similar manner one derives from the geometric Brunn-Minkowski inequality

$$\mathrm{vol}_n(A+B)^{\frac{1}{n}} \geq \mathrm{vol}_n(A)^{\frac{1}{n}} + \mathrm{vol}_n(B)^{\frac{1}{n}} \tag{1.5}$$

the classical isoperimetric inequality

$$H_{n-1}(\partial A) \geq n\omega_n^{\frac{1}{n}}\, \mathrm{vol}_n(A)^{\frac{n-1}{n}}, \tag{1.6}$$

relating the volume of a set A in $\mathbb{R}^n$ to the size of its boundary ∂A. Here ω_n denotes the volume of the unit ball B_1 in $\mathbb{R}^n$, and H_{n-1} stands for the Hausdorff measure of dimension $n-1$. Indeed, applying (1.5) with $B = \varepsilon B_1$ and letting $\varepsilon \to 0$, we arrive at (1.6). In view of this remarkable analogy, Dembo, Costa and Thomas [14] introduced the terminology "isoperimetric inequality for entropies" when speaking about (1.4), which we extend to all relations of the type (1.3) under the name "entropic isoperimetric inequalities".

On the other hand, Carlen [9] noticed that (1.4) is actually equivalent to the logarithmic Sobolev inequality of Gross [16], cf. also [7]. It may be written in the form of the Sobolev-type inequality with respect to the Lebesgue measure

$$\int |f|^2 \log |f|^2 \leq \frac{n}{2} \log \Big[\frac{2}{\pi n e} \int |\nabla f|^2\Big], \quad \int |f|^2 = 1, \tag{1.7}$$

in the class of all smooth f on $\mathbb{R}^n$ subject to the L^2-norm constraint, with gaussian functions playing an extremal role.

Afterwards, in their study of the hypercontractivity properties of non-linear diffusion equations, Del Pino, Dolbeault [12] and Gentil [15] derived a more general L^s-Euclidean logarithmic Sobolev inequality

$$\int |f|^s \log |f|^s \leq \frac{n}{s} \log \Big[L_{s,n} \int |\nabla f|^s\Big], \quad \int |f|^s = 1, \tag{1.8}$$

for $s > 1$ with certain (explicit) constants $L_{s,n}$. They also showed that an equality is achieved for $1 < s < n$, if and only if $f(x)$ is a multiple of the function $\exp\{-c\,|x-a|^{s^*}\}$ with arbitrary $c > 0$ and $a \in \mathbb{R}^n$ (where $s^* = \frac{s}{s-1}$ is the conjugate power, cf. [13]). Starting from this result, in analogy with the equivalence between (1.4) and (1.7), Kitsos and Tavoularis [18] recognized (1.8) as the entropic isoperimetric inequality (1.3) for parameters $\alpha = 1$ and $s > 1$ (however, under the unnecessary condition $s < n$).

The case $s = 1$ may also be included in (1.8). This was earlier shown by Beckner [2], who derived from the isoperimetric inequality (1.6) a logarithmic Sobolev inequality

$$\int |f| \log |f| \leq n \log \Big[L_{1,n} \int |\nabla f|\Big], \quad \int |f| = 1, \tag{1.9}$$

with optimal constant $L_{1,n} = 1/(n\omega_n^{\frac{1}{n}})$. Equality here is attained asymptotically on multiplies of the indicator functions of Euclidean balls in $\mathbb{R}^n$, similarly to the extremal property of balls in (1.6). Hence, applying (1.9) with $f = a\,1_A$, we return to (1.6), which means that (1.9) represents a functional form of the isoperimetric inequality. Being restricted to non-negative functions, this inequality does not lose generality and may be rewritten in terms of the entropy power and the total variation norm as

$$N(X)\, I_1(X)^2 \geq \frac{1}{L_{1,n}^2}$$

for random vectors X with densities $p = f$. Therefore, the inequality (1.3) with parameters $\alpha = s = 1$ and with the optimal constant $c_{1,1,n} = n^2 \omega_n^{2/n}$ represents an equivalent entropic version of the isoperimetric inequality (1.6).

As for general values of α, the family of the entropic isoperimetric inequalities

$$N_\alpha(X)\, I(X) \geq c_{\alpha,n}, \tag{1.10}$$

which corresponds to (1.3) in the special case $s = 2$, has been recently discussed in [6]. As was raised there, the following two natural questions are of most interest:

Question 1. Given n and $s \geq 1$, for which range $\mathfrak{A}_{s,n}$ of the values of α does (1.3) hold with some positive constant?
Question 2. What is the value of the optimal constant $c_{\alpha,s,n}$ and can the extremizers in (1.3) be described?

By the monotonicity of N_α with respect to α, the function $\alpha \mapsto c_{\alpha,s,n}$ is non-increasing. Hence, the range in Question 1 takes necessarily the form

$$\mathfrak{A}_{s,n} = [0, \alpha_{s,n}) \quad \text{or} \quad \mathfrak{A}_{s,n} = [0, \alpha_{s,n}]$$

for some critical value $\alpha_{s,n} \in [0, \infty]$. Similarly, we can include $s = \infty$ in our investigation as a limiting case. Here we prove the following theorem that generalizes [6, Theorem 1.1] and answers Question 1.

Theorem 1.1. *We have*

$$\mathfrak{A}_{s,n} = \begin{cases} [0,\infty] & \text{for } n = 1,\ s \in [1,\infty] \\ [0, \frac{n}{n-s}] & \text{for } n \geq 2,\ s \in [1,n) \\ [0,\infty) & \text{for } n \geq 2,\ s = n \\ [0,\infty] & \text{for } n \geq 2,\ s > n. \end{cases}$$

In our analysis of Questions 1-2, it will be convenient to give an equivalent formulation of (1.3) in terms of functional inequalities. However, in contrast with (1.7)-(1.9) for the parameter $\alpha = 1$, a different class of analytic inequalities should be considered when $\alpha \neq 1$. Namely, using the substitution $p = f^s / \int f^s$ for f non-negative, we have

$$N_\alpha(X) = \Big(\int f^{\alpha s}\Big)^{-\frac{2}{n(\alpha-1)}} \Big(\int f^s\Big)^{\frac{2\alpha}{n(\alpha-1)}}$$

and

$$I_s(X) = s^s \int |\nabla f|^s \Big/ \int f^s.$$

Therefore, provided that f^s is integrable, (1.3) can be equivalently reformulated as a homogeneous analytic inequality

$$\Big(\int f^{\alpha s}\Big)^{\frac{1}{n(\alpha-1)}} \leq \frac{s}{\sqrt{c_{\alpha,s,n}}} \Big(\int |\nabla f|^s\Big)^{\frac{1}{s}} \Big(\int f^s\Big)^{\frac{\alpha}{n(\alpha-1)} - \frac{1}{s}}. \tag{1.11}$$

Using standard density arguments (the reader may find details in the case $s = 2$ in Section 5 of [6]), we can assume in (1.11) that the functions f are smooth enough and compactly supported. Notice however that, when speaking about extremizers, the function f should be allowed to belong to the larger Sobolev space $W_1^s(\mathbb{R}^n) = \{f \in L^s : |\nabla f| \in L^s\}$ where the gradients are understood in a weak sense.

Observe that, if f is Lipschitz then $|\nabla|f|| \leq |\nabla f|$ almost everywhere so that in all inequalities considered in this paper one can restrict to non-negative functions without loss of generality. We will therefore consider non-negative functions all along the paper without any further mention.

Inequalities (1.11) enter the general framework of Gagliardo-Nirenberg's inequalities

$$\Big(\int f^r\Big)^{\frac{1}{r}} \leq \kappa_n(q,r,t)\Big(\int |\nabla f|^q\Big)^{\frac{\theta}{q}}\Big(\int f^t\Big)^{\frac{1-\theta}{t}} \tag{1.12}$$

with $1 \leq q,r,t \leq \infty$, $0 \leq \theta \leq 1$, and $\frac{1}{r} = \theta\,(\frac{1}{q} - \frac{1}{n}) + (1-\theta)\,\frac{1}{t}$. We will make use of the knowledge on Gagliardo-Nirenberg's inequalities to derive information on (1.3).

In the sequel, we denote by $\|f\|_r = (\int |f|^r)^{\frac{1}{r}}$ the L^r-norm of f with respect to the Lebesgue measure on $\mathbb{R}^n$ (and use this functional also in the case $0 < r < 1$).

2. Nagy's theorem

In the next three sections we focus on dimension $n = 1$, in which case the entropic isoperimetric inequality (1.3) takes the form

$$N_\alpha(X)\, I_s(X)^{\frac{2}{s}} \geq c_{\alpha,s,1} \tag{2.1}$$

for the Rényi entropy

$$N_\alpha(X) = \Big(\int p(x)^\alpha dx\Big)^{-\frac{2}{\alpha-1}}$$

and the Fisher information of order $s \geq 1$

$$I_s(X) = \int_{p(x)>0} \frac{p'(x)^s}{p(x)^{s-1}}\, dx = s^s \int \Big(\frac{d}{dx}\, p(x)^{\frac{1}{s}}\Big)^s\, dx.$$

Our basic functional space is the collection of all (locally) absolutely continuous functions on the real line whose derivatives are understood in the Radon-Nikodym sense. Such functions are almost everywhere differentiable. Since p is non-negative, any point $x \in \mathbb{R}$ such that $p(x) = 0$ is a local minimum, and necessarily $p'(x) = 0$ (as long as p is differentiable at x).

Let us show that (2.1) holds for all $\alpha \in [0,\infty]$, proving the following elementary sub-optimal inequality

$$N_\infty(X) I_s(X)^{\frac{2}{s}} \geq 1. \tag{2.2}$$

Assume that $I_s(X)$ is finite, so that X has a (locally) absolutely continuous density p, thus differentiable almost everywhere, with the property that $p(-\infty) = p(\infty) = 0$. Applying Holder's inequality with dual exponents $\frac{1}{s} + \frac{1}{s'} = 1$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} |p'(y)|\, dy \ &= \ \int_{p(y)>0} \frac{|p'(y)|}{p(y)^{\frac{1}{s'}}}\, p(y)^{\frac{1}{s'}}\, dy \\ &\leq \ \Big(\int_{p(y)>0} \frac{|p'(y)|^s}{p(y)^{\frac{s}{s'}}}\, dy\Big)^{1/s} \Big(\int_{p(y)>0} p(y)\, dy\Big)^{1/s'} \ = \ I_s(X)^{\frac{1}{s}}.\end{aligned}$$

It follows that p has a bounded total variation not exceeding $I_s(X)^{\frac{1}{s}}$, so $p(x) \leq I_s(X)^{\frac{1}{s}}$ for every $x \in \mathbb{R}$. This amounts to (2.2) according to (1.2) for $n = 1$.

From (2.2) we deduce that $c_{\alpha,s,1} \geq 1$ for all $\alpha \in [0,\infty]$ and all $s \geq 1$. In order to derive the best possible constant $c_{\alpha,s,1}$, we will make use of a result due to Nagy.

According to (1.11), the family (2.1) takes now the form

$$\int f^{\alpha s} \leq \Big(\frac{s^2}{c_{\alpha,s,1}}\Big)^{\frac{\alpha-1}{2}} \Big(\int |f'|^s\Big)^{\frac{\alpha-1}{s}} \Big(\int f^s\Big)^{\frac{\alpha(s-1)+1}{s}} \tag{2.3}$$

when $\alpha > 1$, and

$$\int f^s \leq \Big(\frac{s^2}{c_{\alpha,s,1}}\Big)^{\frac{(1-\alpha)s}{2(\alpha(s-1)+1)}} \Big(\int |f'|^s\Big)^{\frac{1-\alpha}{\alpha(s-1)+1}} \Big(\int f^{\alpha s}\Big)^{\frac{s}{\alpha(s-1)+1}} \tag{2.4}$$

when $\alpha \in (0,1)$.

In fact, these two families of inequalities can be seen as sub-families of the following one, studied by Nagy [22],

$$\int f^{\gamma+\beta} \leq D\Big(\int |f'|^p\Big)^{\frac{\beta}{pq}} \Big(\int f^\gamma\Big)^{1+\frac{\beta(p-1)}{pq}}$$

with

$$p > 1, \quad \beta,\gamma > 0, \quad q = 1 + \frac{\gamma(p-1)}{p}, \tag{2.5}$$

and some constants $D = D_{\gamma,\beta,p}$ depending on γ, β and p, only. For such parameters, introduce the functions $y_{p,\gamma} = y_{p,\gamma}(t)$ defined for $t \geq 0$ by

$$y_{p,\gamma}(t) = \begin{cases} (1+t)^{\frac{p}{p-\gamma}} & \text{if } p < \gamma, \\ e^{-t} & \text{if } p = \gamma, \\ (1-t)^{\frac{p}{p-\gamma}}\, 1_{[0,1]}(t) & \text{if } p > \gamma. \end{cases}$$

To involve the parameter β, define $y_{p,\gamma,\beta}$ implicitly as follows. Put $y_{p,\gamma,\beta}(t) = u$, $0 \leq u \leq 1$, with

$$t = \int_u^1 \Big(r^\gamma(1-r^\beta)\Big)^{-\frac{1}{p}} dr$$

if $p \leq \gamma$. If $p > \gamma$, then $y_{p,\gamma,\beta}(t) = u$, $0 \leq u \leq 1$, is the solution of the above equation for

$$t \leq t_0 = \int_0^1 \Big(r^\gamma(1-r^\beta)\Big)^{-\frac{1}{p}} dr,$$

and $y_{p,\gamma,\beta}(t) = 0$ for all $t > t_0$. With these notations, Nagy established the following result.

Theorem 2.1. *Under* (2.5)*, for any (locally) absolutely continuous function* $f : \mathbb{R} \to \mathbb{R}$*,*

(i)

$$\|f\|_\infty \leq \Big(\frac{q}{2}\Big)^{\frac{1}{q}} \Big(\int |f'|^p\Big)^{\frac{1}{pq}} \Big(\int f^\gamma\Big)^{\frac{p-1}{pq}}. \tag{2.6}$$

Moreover, the extremizers take the form $f(x) = a y_{p,\gamma}(|bx+c|)$ *with* a, b, c *constants* ($b \neq 0$)*.*

(ii)

$$\int f^{\beta+\gamma} \leq \Big(\frac{q}{2}\, H\Big(\frac{q}{\beta}, \frac{p-1}{p}\Big)\Big)^{\frac{\beta}{q}} \Big(\int |f'|^p\Big)^{\frac{\beta}{pq}} \Big(\int f^\gamma\Big)^{1+\frac{\beta(p-1)}{pq}} \tag{2.7}$$

where

$$H(u,v) = \frac{\Gamma(1+u+v)}{\Gamma(1+u)\,\Gamma(1+v)} \Big(\frac{u}{u+v}\Big)^u \Big(\frac{v}{u+v}\Big)^v, \quad u,v \geq 0.$$

Moreover, the extremizers take the form $f(x) = a y_{p,\gamma,\beta}(|bx+c|)$ *with* a,b,c *constants* ($b \neq 0$)*.*

Here, Γ denotes the Gamma function, and we use the convention $H(u,0) = H(0,v) = 1$ for $u,v \geq 0$. It was mentioned by Nagy that H is monotone in each variable. Moreover, since $H(u,1) = (1+\frac{1}{u})^{-u}$ is between 1 and $\frac{1}{e}$, one has $1 > H(u,v) > (1+\frac{1}{u})^{-u} > \frac{1}{e}$ for all $0 < v < 1$. This gives a two-sided bound

$$1 \geq H\Big(\frac{q}{\beta}, \frac{p-1}{p}\Big) > \Big(1+\frac{\beta}{q}\Big)^{-\frac{q}{\beta}} > \frac{1}{e}.$$

3. One dimensional isoperimetric inequalities for entropies

The inequalities (2.3) and (2.4) correspond to (2.7) with parameters

$$p = \gamma = q = s,\ \ \beta = s(\alpha-1) \ \text{ in the case } \alpha > 1$$

and

$$p = s,\ \beta = s(1-\alpha),\ \gamma = s\alpha,\ q = 1+\alpha(s-1) \ \text{ in the case } \alpha \in (0,1),$$

respectively. Hence, as a corollary from Theorem 2.1, we get the following statement which solves Question 2 when $n = 1$. Note that, by Theorem 2.1, the extremal distributions (their densities p) in (2.1) are determined in a unique way up to non-degenerate affine transformations of the real line. So, it is sufficient to indicate just one specific extremizer for each admissible collection of the parameters. Recall the definition of the optimal constants $c_{\alpha,s,1}$ from (2.1) and the definition of $y_{p,\gamma,\beta}$ before Theorem 2.1. Then, for simplicity of notation we set

$$G_{s,\alpha}(t) = \begin{cases} y^s_{s,s,s(\alpha-1)}(t) & \text{if } \alpha > 1 \\ y^s_{s,\alpha s,s(1-\alpha)}(t) & \text{if } 0 < \alpha < 1 \end{cases}, \qquad t > 0.$$

Theorem 3.1. (i) *In the case* $\alpha = \infty$*, we have, for all* $s \geq 1$*,*

$$c_{\infty,s,1} = 4.$$

Moreover, the density $p(x) = \frac{1}{2}\, e^{-|x|}$ ($x \in \mathbb{R}$) *of the two-sided exponential distribution represents an extremizer in* (2.1)*.*

(ii) *In the case* $1 < \alpha < \infty$ *and* $s > 1$*, we have*

$$c_{\alpha,s,1} = \bigg(2\, \frac{(s-1)^{\frac{1}{s}} (1+\alpha(s-1))^{\frac{s-\alpha+1}{s(\alpha-1)}}}{s^{\frac{1}{\alpha-1}} (\alpha-1)^{\frac{s-1}{s}}}\, \frac{\Gamma(\frac{1}{\alpha-1})\Gamma(\frac{s-1}{s})}{\Gamma(\frac{1+\alpha(s-1)}{s(\alpha-1)})}\bigg)^2,$$

and $p(x) = aG_{s,\alpha}(|x|)$ *with a normalization constant* a *is an extremizer in* (2.1)*.*

(iii) *In the case* $0<\alpha<1$ *and* $s>1$,

$$c_{\alpha,s,1}=\bigg(2\,\frac{s^{\frac{\alpha}{1-\alpha}}(s-1)^{\frac{1}{s}}}{(1+\alpha(s-1))^{\frac{1+\alpha(s-1)}{s(1-\alpha)}}(1-\alpha)^{\frac{s-1}{s}}}\,\frac{\Gamma(\frac{1+\alpha(s-1)}{s(1-\alpha)})\Gamma(\frac{s-1}{s})}{\Gamma(\frac{1}{1-\alpha})}\bigg)^2,$$

and $p(x)=aG_{s,\alpha}(|x|)$ *with a normalization constant* a *is an extremizer in* (2.1).

Proof of Theorem 3.1. When $\alpha=\infty$ as in the case (i), (2.3) with $\int|f|^s=1$ becomes

$$\|f\|_\infty\le\Big(\frac{s^2}{c_{\infty,s,1}}\Big)^{\frac{1}{2s}}\Big(\int|f'|^s\Big)^{\frac{1}{s^2}}.$$

This corresponds to (2.6) with parameters $p=q=\gamma=s$. Therefore, Item (i) of Theorem 2.1 applies (for $s>1$, the case $s=1$ follows by continuity) and leads, when $\int|f|^s=1$, to

$$\|f\|_\infty\le\Big(\frac{s}{2}\Big)^{\frac{1}{s}}\Big(\int|f'|^s\Big)^{\frac{1}{s^2}},$$

that is, $c_{\infty,s,1}=4$. Moreover, the extremizers in (2.6) are given by

$$f(x)=ay_{s,s}(|bx+c|)=a\,e^{-|bx+c|},\quad b\neq 0,\ a,c\in\mathbb{R}.$$

But, the extremizers in (2.1) are of the form $p=f^s/\int f^s$ with f an extremizer in (2.6). The desired result then follows after a change of variables.

Next, let us turn to the case (ii), where $1<\alpha<\infty$. Here (2.1) is equivalent to (2.3) and corresponds to (2.7) with $p=\gamma=q=s$ and $\beta=s(\alpha-1)$. Therefore, by Theorem 2.1, $(\frac{s^2}{c_{\alpha,s,1}})^{\frac{\alpha-1}{2}}=\big(\frac{s}{2}H(\frac{1}{\alpha-1},\frac{s-1}{s})\big)^{\alpha-1}$, so that

$$\begin{aligned}
c_{\alpha,s,1}&=\frac{4}{H(\frac{1}{\alpha-1},\frac{s-1}{s})^2}\\
&=4\,\frac{\Gamma(1+\frac{1}{\alpha-1})^2\,\Gamma(1+\frac{s-1}{s})^2}{\Gamma(1+\frac{1}{\alpha-1}+\frac{s-1}{s})^2}\bigg(\frac{\frac{1}{\alpha-1}+\frac{s-1}{s}}{\frac{1}{\alpha-1}}\bigg)^{\frac{2}{\alpha-1}}\bigg(\frac{\frac{1}{\alpha-1}+\frac{s-1}{s}}{\frac{s-1}{s}}\bigg)^{\frac{2(s-1)}{s}}\\
&=4\bigg(\frac{\frac{s-1}{s(\alpha-1)}}{\frac{\alpha(s-1)+1}{s(\alpha-1)}}\bigg)^2\frac{\Gamma(\frac{1}{\alpha-1})^2\Gamma(\frac{s-1}{s})^2}{\Gamma(\frac{\alpha(s-1)+1}{s(\alpha-1)})^2}\Big(\frac{\alpha(s-1)+1}{s}\Big)^{\frac{2}{\alpha-1}}\Big(\frac{\alpha(s-1)+1}{(\alpha-1)(s-1)}\Big)^{\frac{2(s-1)}{s}}
\end{aligned}$$

where we used the identity $\Gamma(1+z)=z\Gamma(z)$. This leads to the desired expression for $c_{\alpha,s,1}$.

As for extremizers, Item (ii) of Theorem 2.1 applies and asserts that the equality cases in (2.3) are reached, up to numerical factors, for functions $f(x)=y_{s,s,s(\alpha-1)}(|bx+c|)$, with $b\neq 0$, $c\in\mathbb{R}$. Similarly to the case (i), the extremizers in (2.1) are of the form $p=f^s/\int f^s$ with f an extremizer in (2.3). Therefore, $p=aG_{s,\alpha}(|bx+c|)$ with some $b\neq 0$, $c\in\mathbb{R}$ and a a normalization constant, as announced.

Finally, let us turn to item (iii), when $\alpha\in(0,1)$. As already mentioned, (2.1) is equivalent to (2.4) and therefore corresponds to (2.7) with $p=s$, $\beta=s(1-\alpha)$, $\gamma=\alpha s$ and $q=1+\alpha(s-1)$. An application of Theorem 2.1 leads to the desired conclusion after some algebra (which we leave to the reader) concerning the explicit value of $c_{\alpha,s,1}$. In addition, the extremizers are of the form

$p(x) = a y_{s,\alpha s,s(1-\alpha)}(|bx+c|)$, with a a normalization constant, $b \neq 0$ and $c \in \mathbb{R}$. This leads to the desired conclusion. $\square$

4. Limiting orders

In this section we use continuity arguments to obtain explicit values of $c_{\alpha,s,1}$ for $\alpha = 0, 1$ in the one dimensional entropic isoperimetric inequality

$$N_\alpha(X)\, I(X)^{\frac{2}{s}} \geq c_{\alpha,s,1}. \tag{4.1}$$

Notice that, contrary to the special case $s = 2$ treated in [6], it is not possible to obtain an explicit expression for $G_{s,\alpha}$ for all $s \geq 1$ and all $\alpha \in [0, \infty]$. We may, however, be able to give some partial results for some very special values of the parameters.

We anticipate on the fact our discussion below will lead to $\mathfrak{A}_{s,1} = [0, \infty]$ for all $s \in [1, \infty]$, thus proving the first part of Theorem 1.1 corresponding to dimension $n = 1$.

The order $\alpha = 0$. The limit in item (iii) of Theorem 3.1 leads to the optimal constant

$$c_{0,s,1} = \lim_{\alpha \to 0} c_{\alpha,s,1} = 4\pi^2 \frac{(s-1)^{\frac{2}{s}}}{\sin^2(\pi/s)}, \qquad s > 1.$$

To see this, we used the fact that $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, for $z \in \mathbb{C} \setminus \mathbb{Z}$. More precisely, we have

$$c_{0,s,1} = \lim_{\alpha \to 0} c_{\alpha,s,1} = \left(2\, \frac{(s-1)^{\frac{1}{s}} \Gamma(\frac{1}{s}) \Gamma(\frac{s-1}{s})}{\Gamma(1)} \right)^2 = 4\pi^2 \frac{(s-1)^{\frac{2}{s}}}{\sin^2(\pi/s)}.$$

Since all explicit expressions are continuous with respect to α and s, the limits of the extremizers in (2.1) for $\alpha \to 0$ represent extremizers in (2.1) for $\alpha = 0$. However to compute $G_{s,0}$ one would need to solve the implicit equation $t = \int_y^1 \frac{dr}{(1-r^s)^{\frac{1}{s}}}$ that seems solvable only for $s = 2$. To be more precise one can express the latter integral in term of the hypergeometric function, but the result is not explicitely invertible, unless $s = 2$.

The order $\alpha = 1$. This case corresponds to Stam's isoperimetric inequality for entropies when $s = 2$, and in this case $c_{1,2,1} = 2\pi e$ (note that this can be deduced using the Stirling formula from the expression of $c_{\alpha,2,1}$ in the limit $\alpha \to 1$). For a generic $s > 1$, we obtain,

$$c_{1,s,1} = \lim_{\alpha \to 1} c_{\alpha,s,1} = \left(2 \Big(\frac{s-1}{s} \Big)^{\frac{1}{s}} \Gamma\Big(\frac{s-1}{s} \Big) e^{\frac{s-1}{s}} \right)^2.$$

Now we explain, at a heuristic level, how to recover the extremizers discovered by Del Pino, Dolbeault and Gentil [13] and described in the introduction. For $s > 1$ and $\alpha > 1$, the extremizers in (4.1) are of the form $ay(|bx+c|)^s$ with $b \neq 0$, a the normalization constant, and with $y(t)$ implicitly defined by the equation

$t = \int_y^1 \frac{dr}{r(1-s^{s(\alpha-1)})^{\frac{1}{s}}}$. Choose $b = 1/(\alpha-1)^{1/s}$. Then y satisfies

$$t = \int_y^1 \frac{(\alpha-1)^{\frac{1}{s}}}{r(1-r^{s(\alpha-1)})^{\frac{1}{s}}}\, dr.$$

In the limit $\alpha \to 1$ we obtain that, up to a multiplicative factor and linear transformation, the function y^s with y being the solution of

$$t = \int_y^1 \frac{1}{r(-s\log r)^{\frac{1}{s}}}\, dr$$

represents an extremizer of (4.1) for $\alpha = 1$ to be rigorous at least one would need that y, that depends on α, converges to some limit (to know that an extremizer exists a priori could be enough)). Changing variable ($u = -s\log r$), we get

$$\int_y^1 \frac{dr}{r(-s\log r)^{\frac{1}{s}}} = \frac{1}{s}\int_0^{-s\log y} \frac{du}{u^{\frac{1}{s}}} = \frac{1}{s-1}(-s\log y)^{\frac{s-1}{s}}.$$

Therefore

$$y(t) = e^{-\frac{1}{s}[(s-1)t]^{\frac{s}{s-1}}},$$

and an extremizer of (4.1) for $\alpha = 1$ is

$$p(x) = ae^{-|bt+c|^{\frac{s}{s-1}}}$$

with $b \neq 0$ and a the normalization constant. In particular, for $s = 2$ this recovers the Gaussian density as extremizer of Stam's isoperimetric inequality.

The order $\alpha = \infty$. From Theorem 3.1, $c_{\infty,s,1} = 4$, for all $s \geq 1$ and the extremizers are of the form

$$p(x) = \frac{b}{2}\, e^{-|bx+c|}, \quad b > 0,\ c \in \mathbb{R}.$$

The order $s = 1$. Taking the limit in the expressions of $c_{\alpha,s,1}$ leads, for any $\alpha > 1$ and any $\alpha \in (0,1)$ and therefore by continuity for any $\alpha \in [0,\infty]$, to the value

$$c_{\alpha,1,1} = 4.$$

Extremizers for $s = 1$ are limit of extremizers for $s > 1$. The implicit equation satisfied by an extremizer y of (4.1) must therefore be of the form $t = \int \frac{dr}{r^\alpha(1-r^{1-\alpha})}$ for $\alpha < 1$ and $t = \int \frac{dr}{r(1-r^{\alpha-1})}$ for $\alpha > 1$. Since both integrals are infinite (and both corresponding integrals are finite for $s > 1$) we conclude that (4.1) has no extremizer for $s = 1$ and all $\alpha \neq 1$ (therefore by continuity for all $\alpha \in [0,\infty]$).

The order $s = 2$. As already mentioned, the case $s = 2$ is studied in depth in [6]. In that case extremizers can be made explicit for all values of $\alpha \in [0,\infty]$. As an illustration, let us mention some examples borrowed from [6] where the reader may find more details and comments.

- For $\alpha = 0$ we have $c_{0,2,1} = 4\pi^2$ and extremizers are

$$p(x) = \frac{2b}{\pi}\cos^2(bx+c)1\!\!1_{[-\frac{\pi}{2},\frac{\pi}{2}]}(bx+c), \quad b>0,\ c\in\mathbb{R}.$$

- For $\alpha = 2$ we have $c_{2,2,1} = 12$ and extremizers take the form

$$p(x) = \frac{b}{2\cosh^2(bx+c)}, \quad b>0,\ c\in\mathbb{R}.$$

- For $\alpha = 3$ we have $c_{3,2,1} = \pi^2$ and

$$p(x) = \frac{b}{\pi\cosh(bx+c)}, \quad b>0,\ c\in\mathbb{R}.$$

The order $s = \infty$. Taking the limit $s \to \infty$ in Theorem 3.1, we get, for $1 < \alpha < \infty$

$$c_{\alpha,\infty,1} = \lim_{s\to\infty} c_{\alpha,s,1} = \left(2\,\frac{\alpha^{\frac{1}{\alpha-1}}}{\alpha-1}\,\frac{\Gamma(\frac{1}{\alpha-1})\Gamma(1)}{\Gamma(\frac{\alpha}{\alpha-1})}\right)^2 = 4\,\alpha^{\frac{2}{\alpha-1}}$$

where for the last equality we used that $\frac{1}{\alpha-1}\Gamma(\frac{1}{\alpha-1}) = \Gamma(1+\frac{1}{\alpha-1}) = \Gamma(\frac{\alpha}{\alpha-1})$ and $\Gamma(1) = 1$.

Similarly, for $\alpha \in (0,1)$ we have

$$c_{\alpha,\infty,1} = \lim_{s\to\infty} c_{\alpha,s,1} = \left(2\,\frac{1}{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}\,\frac{\Gamma(\frac{\alpha}{1-\alpha})\Gamma(1)}{\Gamma(\frac{1}{1-\alpha})}\right)^2 = 4\,\alpha^{\frac{2}{\alpha-1}}.$$

In particular, $c_{1,\infty,1} = 4e^2$ as one can alternatively obtain from the case $\alpha = 1$ above.

For extremizers in (4.1) when $s = \infty$ and $\alpha > 1$, applying the monotone convergence theorem for decreasing sequence (note that, for $s = 2$, say, $\int_y^1 \frac{dr}{r(1-r^{s(\alpha-1)})^{\frac{1}{s}}} < \infty$), the limit of the implicit equation is

$$t = \int_y^1 \frac{dr}{r} = -\log(y).$$

This suggests that the extremizers in (4.1) are $p(x) = \frac{b}{2}e^{-|bx+c|}$, with $b > 0$ and $c \in \mathbb{R}$. This is only at a heuristic level since the extremizers in (4.1) are of the form $p(x) = y^s/\int y^s$ with $s \to \infty$ and the above argument only says that, in the limit $s \to \infty$, $y(x) = \frac{b}{2}e^{-|bx+c|}$, not p. Now one can observe that $y^s/\int y^s$ has morally the same shape as $be^{-|bx+c|}$ by changing the constants b, c. In fact, one can check by hand (details are left to the reader) that the densities $p(x) = \frac{b}{2}e^{-|bx+c|}$, with $b > 0$ and $c \in \mathbb{R}$, are indeed extremizers in (4.1), however not excluding the existence of other extremizers.

5. Dimension $n = 2$ and higher: analysis of the range $\mathfrak{A}_{s,n}$ (Proof of Theorem 1.1)

In this section we consider Question 1 in the introduction about the entropic isoperimetric inequality (1.3) in dimension $n \geq 2$ and prove Theorem 1.1.

First, assuming that $1 \le s \le n$, let us rewrite (1.11) for three natural regions, namely

$$\Big(\int f^s\Big)^{\frac{1}{s}} \le \Big(\frac{s^2}{c_{\alpha,s,n}}\Big)^{\frac{\theta}{2}} \Big(\int |\nabla f|^s\Big)^{\frac{\theta}{s}} \Big(\int f^{\alpha s}\Big)^{\frac{1-\theta}{\alpha s}}, \qquad \alpha \in (0,1), \tag{5.1}$$

with $\theta = \frac{n(1-\alpha)}{\alpha s + n(1-\alpha)}$,

$$\Big(\int f^{\alpha s}\Big)^{\frac{1}{\alpha s}} \le \Big(\frac{s^2}{c_{\alpha,s,n}}\Big)^{\theta} \Big(\int |\nabla f|^s\Big)^{\frac{\theta}{s}} \Big(\int f^{s}\Big)^{\frac{1-\theta}{s}}, \qquad 1 < \alpha \le \frac{n}{n-s}, \tag{5.2}$$

with $\theta = \frac{n(\alpha-1)}{\alpha s}$, and

$$\Big(\int f^{\alpha s}\Big)^{\frac{\theta}{\alpha s}} \Big(\int f^{s}\Big)^{\frac{1-\theta}{s}} \le \frac{s}{\sqrt{c_{\alpha,s,n}}} \Big(\int |\nabla f|^s\Big)^{\frac{1}{s}}, \qquad \alpha > \frac{n}{n-s}, \tag{5.3}$$

with $\theta = \frac{\alpha s}{n(\alpha-1)}$. Note that $\theta \in (0,1)$ in all cases. By the convention, $\frac{n}{n-s} = \infty$ when $s = n$, in which case (5.3) does not exist, while the range in (5.2) is $1 < \alpha \le \infty$.

We distinguish between four different cases.

5.1. $1 < s < n$. We employ the Sobolev inequality, which corresponds to the Gagliardo-Nirenberg inequality (1.12) in the limiting case $\theta = 1$, with $q = s$. Namely, we have

$$\Big(\int f^{\frac{ns}{n-s}}\Big)^{\frac{n-s}{sn}} \le S_{n,s} \Big(\int |\nabla f|^s\Big)^{\frac{1}{s}} \tag{5.4}$$

with best constant

$$S_{n,s} = \frac{1}{\sqrt{\pi}\, n^{\frac{1}{s}}} \Big(\frac{s-1}{n-s}\Big)^{1-\frac{1}{s}} \Big(\frac{s}{2(s-1)}\, \frac{\Gamma(n)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{s})\Gamma(n(1-\frac{1}{s}))}\Big)^{\frac{1}{n}}$$

(cf. [1, 25]). Moreover, all extremizers in (5.4) have the form

$$f(x) = \frac{a}{(1 + b\,|x-x_0|^{\frac{s}{s-1}})^{\frac{n}{s}-1}} \tag{5.5}$$

with arbitrary $a \in \mathbb{R}$, $b > 0$, $x_0 \in \mathbb{R}^n$. This Sobolev inequality reflects the best possible embedding in the sense that the exponent on the left-hand side cannot be improved: For any $p > \frac{ns}{n-s}$, there exists a function in $W_1^s(\mathbb{R}^n)$ that does not belong to $L^p(\mathbb{R}^n)$. Also, for $s \ge n$ it is impossible to replace $S_{n,s}$ by a finite constant in (5.4), so that $s < n$ is necessary.

Recall that (5.1)-(5.2) enter the general framework of the Gagliardo-Nirenberg inequality (1.12) (under the restriction $\alpha s \ge 1$ for (5.1)). We observe also that (5.3) does not hold; otherwise this inequality would imply that any function in $W_1^s(\mathbb{R}^n)$ belongs to $L^{\alpha s}(\mathbb{R}^n)$ as well. But this contradicts the optimal Sobolev embeddings in view of $\alpha s > \frac{ns}{n-s}$.

From the above discussion we conclude that for $n \ge 2$ and $s \in (1,n)$, the inequality (1.3) holds true for any $\alpha \in (1/s, 1)$ and $\alpha \in (1, \frac{n}{n-s}]$, and it does not hold for $\alpha > \frac{n}{n-s}$. By the monotonicity argument, it follows that for $s \in (1,n)$ and $n \ge 2$,

$$\mathfrak{A}_{s,n} = \Big[0, \frac{n}{n-s}\Big].$$

5.2. $s = 1$. In this case the Sobolev inequality (5.4) is reduced to

$$\left(\int f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq S_{n,1} \int |\nabla f| \tag{5.6}$$

with best constant

$$S_{n,1} = \frac{\Gamma(\frac{n}{2}+1)^{\frac{1}{n}}}{n\sqrt{\pi}}.$$

Inequality (5.6) is a functional version of the classical isoperimetric inequality (1.6), and an equality is attained in the asymptotic sense when f approaches the indicator function of a ball in $\mathbb{R}^n$. The exponent on the left hand side of (5.6) cannot be improved.

Reasoning in similar way than for $1 < s < n$, we observe that (5.1) does not enter the framework of the Gagliardo-Nirenberg Inequality (1.12), while (5.2) does, and (5.3) cannot hold. As a consequence, using again the monotonicity argument, we conclude that, when $s = 1$ and $n \geq 2$,

$$\mathfrak{A}_{1,n} = \left[0, \frac{n}{n-1}\right].$$

5.3. $s = n$. In this case, (5.4) should be replaced by Moser-Trudinger's inequality (see *e.g.* [23] for the historical presentation and references). We observe that (5.1) and (5.2) enter the framework of Gagliardo-Nirenberg inequality (1.12) (for $\alpha \in [1/n, 1)$ for (5.1)). Therefore, by monotonicity, (1.3) holds for all $\alpha \in [0, \infty)$, and we only need to analyze separately the case $\alpha = \infty$. For $s = n$ and $\alpha = \infty$, which corresponds to (5.2) in the limit $\alpha \to \infty$, the inequality reads

$$\|f\|_\infty \leq C \left(\int |\nabla f|^n\right)^{\frac{1}{n}} \tag{5.7}$$

with $C = n/\sqrt{c_{\infty,n,n}}$. However (5.7) cannot hold with any constant D as shown in Example 1.1.1 in [23]. Therefore,

$$\mathfrak{A}_{n,n} = [0, \infty).$$

5.4. $s > n$. As mentioned in the introduction, by monotonicity, the range $\mathfrak{A}_{s,n}$ must take the form $[0, \alpha_{s,n})$ or $[0, \alpha_{s,n}]$. In particular, for a given s, if (1.3) holds for some $\alpha_0 \geq 0$, it holds for $\alpha \leq \alpha_0$. Let see that, when $s > n$, (1.3) holds for $\alpha = \infty$, thus proving that $\mathfrak{A}_{s,n} = [0, \infty]$ as stated in Theorem 1.1. To that aim, take the limit $\alpha \to \infty$ in (1.11). We then obtain

$$\|f\|_\infty^{\frac{s}{n}} \leq D \left(\int |\nabla f|^s\right)^{\frac{1}{s}} \left(\int f^s\right)^{\frac{1}{n}-\frac{1}{s}} \tag{5.8}$$

with $D = s/\sqrt{c_{\infty,s,n}}$. This corresponds to the Gagliardo-Nirenberg inequality (1.12) with $r = \infty$, $q = t = s$, for which we have $\theta = \frac{n}{s} \in (0, 1)$ when $s > n$. Therefore, the above inequality holds (with $D = s/\sqrt{c_{\infty,s,n}} = \kappa_n(s, \infty, s)^{\frac{s}{n}} < \infty$) and $\mathfrak{A}_{s,n} = [0, \infty]$ as announced.

6. Extremizers in the isoperimetric inequalities for entropies (1.3) when $n \geq 2$, analysis of Question 2.

In this section we consider Question 2 in the introduction in dimension $n \geq 2$.

As already mentioned, (1.11) enters the framework of the Gagliardo-Nirenberg inequality (1.12). The best constants and extremizers in (1.12) are not known for all admissible parameters. The most recent paper on this topic is due to Liu and Wang [21] (see references therein and historical comments). The case $q = t = 2$ in (1.12) goes back to Weinstein [28] who related the best constants to the solutions of non-linear Schrödinger equations.

We present now part of the results of [21] that are useful for us. Since all the inequalities of interest for us deal with the L^s-norm of the gradient only, we may restrict ourselves to $q = s$ for simplicity, when (1.12) becomes

$$\Big(\int f^r\Big)^{\frac{1}{r}} \leq \kappa_n(s,r,t)\Big(\int |\nabla f|^s\Big)^{\frac{\theta}{s}}\Big(\int f^t\Big)^{\frac{1-\theta}{t}} \tag{6.1}$$

with parameters satisfying $1 \leq r, t \leq \infty$, $0 \leq \theta \leq 1$, and $\frac{1}{r} = \theta(\frac{1}{s} - \frac{1}{n}) + (1-\theta)\,\frac{1}{t}$. This inequality may be restricted to the class of all smooth, compactly supported functions $f \geq 0$ on $\mathbb{R}^n$. Once (6.1) holds in $C_0^\infty(\mathbb{R}^n)$, this inequality is extended by a regularization and density arguments to all Sobolev functions $f \in W_1^s(\mathbb{R}^n)$.

The next statement relates the optimal constant in (6.1) to the solutions of some ordinary non-linear equations. In the sequel, for the range of parameters $1 \leq r, t \leq \infty$, we denote by $u = u_{r,s,t,n}$ (with variable y), the positive decreasing solution of the ordinary non-linear differential equation

$$(|u'|^{s-2}u')' + \frac{n-1}{y}|u'|^{s-2}u' + u^{r-1}\mathbb{1}_{r\neq\infty} = u^{t-1} \tag{6.2}$$

on the positive half-axis. Put

$$\sigma = \begin{cases} \frac{n(s-1)+s}{n-s} & \text{if } n > s, \\ \infty & \text{if } n \leq s. \end{cases}$$

We denote by $|x|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$.

Theorem 6.1 ([21]). *The following holds.*

(a) *In the range $s > \frac{2n}{n+2}$, $t \in (1,\sigma)$ and $r \in (t, \sigma+1)$,*

$$\kappa_n(s,r,t) = \theta^{-\frac{\theta}{s}}(1-\theta)^{\frac{\theta}{s}-\frac{1}{r}} M_t^{-\frac{\theta}{n}}, \quad M_t = \int_{\mathbb{R}^n} u^t(|x|)\,dx,$$

where the functions $u = u(y)$ are defined for $y \geq 0$ as follows.

(a-i) *If $t < s$, then u is the unique positive decreasing solution to the equation (6.2) in $0 < y < y_0$ (for some y_0), satisfying $u'(0) = 0$, $u(y_0) = u'(y_0) = 0$, and $u(y) = 0$ for all $y \geq y_0$.*

(a-ii) *If $t \geq s$, then u is the unique positive decreasing solution to (6.2) in $y > 0$, satisfying $u'(0) = 0$ and $\lim_{y\to\infty} u(y) = 0$.*

(b) *In the range $s > n$, $t \geq 1$ and $r = \infty$,*

$$\kappa_n(s,\infty,t) = \theta^{-\frac{\theta}{s}}(1-\theta)^{\frac{\theta}{s}} M_t^{-\frac{\theta}{n}}, \quad M_t = \int_{\mathbb{R}^n} u^t(|x|)\,dx,$$

where the functions $u = u(y)$ are defined for $y \geq 0$ as follows.

(b-i) *If $t < s$, then u is the unique positive decreasing solution to the equation (6.2) in $0 < y < y_0$ (for some y_0), satisfying $u(0) = 1$, $u(y_0) = u'(y_0) = 0$, and $u(y) = 0$ for all $y \geq y_0$.*

(b-ii) If $t \geq s$, then u is the unique positive decreasing solution to (6.2) in $y > 0$, satisfying $u(0) = 1$ and $\lim_{y\to\infty} u(y) = 0$.

Moreover, the extremizers in (6.2) exist and have the form $f(x) = au(|bx + c|)$ with $a \in \mathbb{R}$, $b \neq 0$, $c \in \mathbb{R}^n$.

Note that (6.1) corresponds to Gagliardo-Nirenberg's inequality (1.11) with $r = \alpha s$, $t = s$ and $\theta = \frac{n(\alpha-1)}{\alpha s}$ for $\alpha > 1$, while (1.11) with $\alpha \in [\frac{1}{s}, 1)$ corresponds to (6.1) with $r = s$, $t = \alpha s$ and $\theta = \frac{n(1-\alpha)}{\alpha s + n(1-\alpha)}$. By (5.2) we therefore conclude that

$$\begin{aligned}
\kappa_n(s, \alpha s, s) &= (s^2/c_{\alpha,s,n})^{\frac{n(\alpha-1)}{2\alpha s}} = (s^2/c_{\alpha,s,n})^{\frac{\theta}{2}} \quad \text{when } \alpha \in \mathfrak{A}_{s,n} \text{ and } \alpha > 1,\\
\kappa_n(s, s, \alpha s) &= (s^2/c_{\alpha,s,n})^{\frac{n(1-\alpha)}{2(\alpha s + n(1-\alpha))}} = (s^2/c_{\alpha,s,n})^{\frac{\theta}{2}} \quad \text{when } \alpha \in [1/s, 1).
\end{aligned}$$

Together with Liu-Wang's theorem, we get the following corollary, where we put as before

$$M_t = \int_{\mathbb{R}^n} u^t(|y|)\, dy$$

and

$$s_0 = s_0(n, \alpha) = \max\Big\{\frac{-1 + \sqrt{1 + 4n}}{2}; \frac{n(\alpha - 1)}{\alpha}\Big\},$$

$$s_1 = s_1(n, \alpha) = \max\Big\{\frac{-1 - n(1-\alpha) + \sqrt{(1 + n(1-\alpha))^2 + 4\alpha n}}{2\alpha}; \frac{2n}{n+2}; \frac{1}{\alpha}\Big\}.$$

Corollary 6.2. *Let $n \geq 2$. Then, the following holds.*

(i) *For any $\alpha \in \mathfrak{A}_{s,n}$, $\alpha > 1$, and any $s > s_0$, we have*

$$c_{\alpha,s,n} = s^2 (n(\alpha-1))^{\frac{2}{s}} (\alpha s)^{-\frac{2}{n(\alpha-1)}} (\alpha s - n(\alpha-1))^{\frac{2(s-n(\alpha-1))}{sn(\alpha-1)}} M_s^{\frac{2}{n}},$$

where M_s is defined for the unique positive decreasing solution $u(y)$ on $(0, \infty)$ to the equation

$$(|u'|^{s-2}u')' + \frac{n-1}{y}|u'|^{s-2}u' + u^{\alpha s-1} = u^{s-1}$$

with $u'(0) = 0$ and $\lim_{y\to\infty} u(y) = 0$.

(ii) *For $\alpha = \infty$ and $s > n$, we have*

$$c_{\infty,s,n} = s^2 \left(\frac{n}{s-n}\right)^{\frac{2}{s}} M_s^{\frac{2}{n}},$$

where M_s is defined for the unique positive decreasing solution $u(y)$ on $(0, \infty)$ to the equation

$$(|u'|^{s-2}u')' + \frac{n-1}{y}|u'|^{s-2}u' = u^{s-1}$$

with $u(0) = 1$ and $\lim_{y\to\infty} u(y) = 0$.

(iii) *For any $\alpha \in (0,1)$ and $s > s_1$, we have*

$$c_{\alpha,s,n} = s^2 \left(n(1-\alpha)\right)^{\frac{2}{s}} (\alpha s)^{\frac{2\alpha}{n(1-\alpha)}} \left(\alpha s + n(1-\alpha)\right)^{-\frac{2(\alpha s+n(1-\alpha))}{ns(1-\alpha)}} M_{\alpha s}^{\frac{2}{n}},$$

where $M_{\alpha s}$ is defined for the unique positive decreasing solution $u(y)$ to

$$(|u'|^{s-2}u')' + \frac{n-1}{y}|u'|^{s-2}u' + u^{s-1} = u^{\alpha s-1}$$

in $0 < y < Y$ with $u'(0) = 0$, $u(Y) = u'(Y) = 0$, and $u(y) = 0$ for all $y \geq Y$.

In all cases, extremizers of (1.3) *are of the form $p(y) = au^s(|by+c|)$, $y \in \mathbb{R}^n$, $b > 0$, $c \in \mathbb{R}^n$ and a the normalization constant (for (i) and (ii), $a = \frac{b}{M}$).*

Proof of Corollary 6.2. The only point to investigate for Item (i) is the range of admissible s. For $\alpha > 1$, (1.11) corresponds to the Gagliardo-Nirenberg inequality (6.1) with $r = \alpha s$ and $t = s$. Therefore Theorem 6.1 applies if the following constraints are satisfied:

$$s > \frac{2n}{n+2}, \quad s \in (1,\sigma) \quad \text{and} \quad s < \alpha s < \sigma + 1$$

with

$$\sigma = \begin{cases} \frac{n(s-1)+s}{n-s} & \text{if } n > s, \\ \infty & \text{if } n \leq s. \end{cases} \tag{6.3}$$

Consider first the case $s < n$ for which $\sigma = \frac{n(s-1)+s}{n-s}$. We observe that the condition $s \in (1,\sigma)$ amounts to $s > \frac{-1+\sqrt{1+4n}}{2}$ and $s < \alpha s < \sigma + 1$ to $s > \frac{n(\alpha-1)}{\alpha}$. Since, for any $n \geq 2$,

$$n \geq \frac{-1+\sqrt{1+4n}}{2} \geq \frac{2n}{n+2},$$

the constraints summarize into $s \in (\max(\frac{-1+\sqrt{1+4n}}{2}; \frac{n(\alpha-1)}{\alpha}), n)$.

For $s \geq n$, $\sigma = \infty$ and the constraints reduce to $s > \frac{2n}{n+2}$. All together we get the range $s \in (\max(\frac{-1+\sqrt{1+4n}}{2}; \frac{n(\alpha-1)}{\alpha}), \infty)$ as announced.

Consider now the case $\alpha = \infty$ which corresponds to (1.11) in the limit $\alpha \to \infty$, namely

$$\|f\|_\infty^{\frac{s}{n}} \leq \frac{s}{\sqrt{c_{\infty,s,n}}} \Big(\int |\nabla f|^s\Big)^{\frac{1}{s}} \Big(\int f^s\Big)^{\frac{1}{n}-\frac{1}{s}}. \tag{6.4}$$

This is the Gagliardo-Nirenberg inequality (6.1) with parameters $r = \infty$, $q = t = s$, for which $\theta = \frac{n}{s} \in (0,1)$ when $s > n$. Therefore Item (b-ii) of Theorem 6.1 applies and

$$\Big(\frac{s}{\sqrt{c_{\infty,s,n}}}\Big)^{\frac{n}{s}} = \kappa_n(s,\infty,s) = \Big(\frac{1-\theta}{\theta}\Big)^{\frac{\theta}{s}} M_s^{-\frac{\theta}{n}}.$$

It follows that

$$c_{\infty,s,n} = s^2 \left(\frac{\theta}{1-\theta}\right)^{\frac{2}{s}} M_s^{\frac{2}{n}} = s^2 \Big(\frac{n}{s-n}\Big)^{\frac{2}{s}} M_s^{\frac{2}{n}}$$

as announced.

Now we turn to Item (iii). As for Item (i) the only point to be analyzed is the range of admissible s. For $\alpha < 1$, (1.11) corresponds to the Gagliardo-Nirenberg

Inequality (6.1) with $t = \alpha s$ and $r = s$. Therefore Theorem 6.1 applies if the following constraints are satisfied:

$$s > \frac{2n}{n+2}, \quad \alpha s \in (1, \sigma) \quad \text{and} \quad \alpha s < s < \sigma + 1$$

with σ as in (6.3). When $s < n$, $\sigma = \frac{n(s-1)+s}{n-s}$ we observe that the condition $\alpha s \in (1, \sigma)$ amounts to

$$s > \max\Big\{\frac{1}{\alpha}; \frac{-1 - n(1-\alpha) + \sqrt{(1+n(1-\alpha))^2 + 4\alpha n}}{2\alpha}\Big\}$$

while $\alpha s < s < \sigma + 1$ is always satisfied. The constraints summarize into $s \in (s_1, n)$ (with the convention that the interval is empty if $s_1 \geq n$, which may occur if, for instance, $\alpha \leq 1/n$). When $s \geq n$, $\sigma = \infty$ and the constraints reduce to $s > \max(\frac{2n}{n+2}; \frac{1}{\alpha})$. All together we get the range $s \in (s_1, \infty)$ as announced. $\square$

For some specific values of the parameters, the picture is more complete.

Note first that the case $\alpha = 1$, $s = 2$, which is formally not contained in the results above, is the classical isoperimetry inequality for entropies (1.4). Also, as already mentioned in the introduction, for $\alpha = 1$ and $s > 1$, by means of the Euclidean log-Sobolev inequality [13, 18], extremizers in (1.3) are of the form $p(x) = b \exp\{-c\,|x-a|^{\frac{s}{s-1}}\}$, $a \in \mathbb{R}^n$, $c > 0$ and a the normalisation constant. For $1 < s < n$ such densities are the only extremizers in (1.3), while for $s \geq n$, there might exist other ones.

The next statement deals with the special case $\alpha = \frac{n}{n-s}$ that corresponds to Sobolev's inequality.

Corollary 6.3. *Let $n \geq 2$, $n > s \geq 1$ and $\alpha = \frac{n}{n-s}$. Then*

$$c_{\alpha,s,n} = \begin{cases} \pi s^2 n^{\frac{2}{s}} \left(\frac{n-s}{s-1}\right)^{\frac{2(s-1)}{s}} \left(\frac{2(s-1)}{s} \frac{\Gamma(\frac{n}{s})\Gamma(n(1-\frac{1}{s}))}{\Gamma(n)\Gamma(\frac{n}{2})}\right)^{\frac{2}{n}} & \text{if } s > 1 \\ \pi n^2 \Gamma(\frac{n}{2}+1)^{-\frac{2}{n}} & \text{if } s = 1. \end{cases}$$

(i) *For $n \leq s^2$ or $s = 1$ and $n \geq 2$, (1.3) has no extremizers, i.e. there does not exist any density p for which equality holds in (1.3) with the optimal constant.*

(ii) *For $n > s^2$, the extremizers in (1.3) exist and have the form*

$$p(x) = \frac{a}{(1 + b|x - x_0|^{\frac{s}{s-1}})^{n-s}}, \quad a, b > 0, \ x_0 \in \mathbb{R}^n.$$

Proof of Corollary 6.3. For $\alpha = \frac{n}{n-s}$, (1.11) corresponds to the Gagliardo-Nirenberg inequality (1.12) for $r = \alpha s$, $t = s$ and $\theta = 1$, namely

$$\Big(\int f^{\alpha s}\Big)^{\frac{1}{\alpha s}} \leq \kappa_n(s, \alpha s, s)\Big(\int |\nabla f|^s\Big)^{\frac{1}{s}}.$$

This is the Sobolev inequality (5.4) when $s > 1$ and (5.6) when $s = 1$, with best constant

$$\kappa_n(s, \alpha s, s) = S_{n,s} = \frac{1}{\sqrt{\pi} n^{\frac{1}{s}}} \left(\frac{s-1}{n-s}\right)^{1-\frac{1}{s}} \Big(\frac{s}{2(s-1)} \frac{\Gamma(n)\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{s})\Gamma(n(1-\frac{1}{s}))}\Big)^{\frac{1}{n}}, \qquad s > 1,$$

and

$$\kappa_n(1,\alpha,1) = S_{n,1} = \frac{\Gamma(\frac{n}{2}+1)^{\frac{1}{n}}}{n\sqrt{\pi}}, \qquad s = 1.$$

Since $\kappa_n(s,\alpha s,s) = \frac{s}{\sqrt{c_{\alpha,s,n}}}$, we get the expected explicit expression of the optimal constant $c_{\alpha,s,n}$.

Moreover, as mentioned in Section 5, the only extremizers in (5.4) have the form

$$f(x) = \frac{a}{(1+b\,|x-x_0|^{\frac{s}{s-1}})^{\frac{n}{s}-1}}, \quad a \in \mathbb{R},\ b>0,\ x_0 \in \mathbb{R}^n$$

while (5.6) has no extremizers. Assume that we have an equality in (1.3) for a fixed (probability) density p on $\mathbb{R}^n$. We should assume that the function $f = p^{\frac{1}{s}}$ belongs to $W_1^s(\mathbb{R}^n)$ and f must be of the above form. However, whether or not this function p is integrable depends on the dimension. Using polar coordinates, one immediately realizes that

$$\int \frac{dx}{(1+b\,|x-x_0|^{\frac{s}{s-1}})^{n-s}}$$

has the same behavior as $\int_1^\infty \frac{1}{r^d}dr$ with $d = \frac{n-s^2+s-1}{s-1}$. But, the latter integral converges only if $n > s^2$. ☐

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Sergey G. Bobkov
School of Mathematics, University of Minnesota, Minneapolis, MN, USA
E-mail address: bobko001@umn.edu

Cyril Roberto
MODAL'X, UMR CNRS 9023, CNRS FR2036, Universit 辿 Paris Nanterre
E-mail address: cyril.roberto@math.cnrs.fr