One Garnir to Rule Them All

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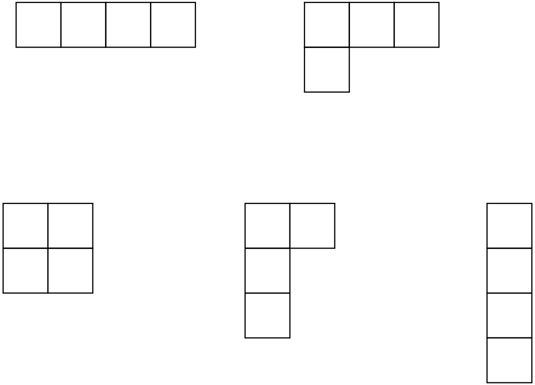
Representations of the Symmetric Group

Irreducible representations of S_n correspond to partitions λ of n



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3. Construct a row polytabloid:

$$e_{\mathbf{t}} = \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} - \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{cc}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} + \begin{array}{cc}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$$

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S^λ has a basis of row polytabloids $e_{\mathbf{t}}$ where the corresponding $\mathbf{t}$ is a *Standard Young Tableau.*

From our motivation, it would be more convenient to realize the Specht Module S^λ as a **quotient** of elements which are naturally **anti-symmetric.**

Column Tabloids

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A Dual Construction

Fulton defines a map

$$\alpha : \text{column tabloids of shape } \lambda \to \text{ Specht module of shape } \lambda$$

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$$\alpha : \left|\begin{array}{c|c} 1 & 3 \\ 2 & 4 \end{array}\right| \mapsto \frac{\overline{\;1 \quad 3\;}}{\underline{\;2 \quad 4\;}} - \frac{\overline{\;2 \quad 3\;}}{\underline{\;1 \quad 4\;}} - \frac{\overline{\;1 \quad 4\;}}{\underline{\;2 \quad 3\;}} + \frac{\overline{\;2 \quad 4\;}}{\underline{\;1 \quad 3\;}}$$

What is $\ker(\alpha)$?

A Dual Straightening Algorithm

Definition (Fulton)

A *dual Garnir relations* is

$$\pi_{c,k}(\mathbf{t}) := \sum [\mathbf{s}]$$

where the sum is over column tabloids obtained by exchanging the top k elements in the $(c+1)^{st}$ column in all possible ways with k elements in the c^{th} column

A Dual Straightening Algorithm

$$\pi_{1,1}\left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array}\right) = \left|\begin{array}{c|c} 4 & 1 \\ 2 & 5 \\ 3 & \end{array}\right| + \left|\begin{array}{c|c} 1 & 2 \\ 4 & 5 \\ 3 & \end{array}\right| + \left|\begin{array}{c|c} 1 & 3 \\ 2 & 5 \\ 4 & \end{array}\right|$$

Theorem (Fulton)

The relations

$$[\mathbf{t}] - \pi_{c,k}(\mathbf{t})$$

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Upshot: The Specht Module S^λ can be realized as a quotient module of the space of column tabloids by dual Garnir relations.

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$$\pi_{1,1}\left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array}\right) = \left|\begin{matrix} \color{red}{4} \\ 2 \\ 3 \end{matrix}\right|\begin{matrix} \color{red}{1} \\ 5 \\ \end{matrix}\right| + \left|\begin{matrix} 1 \\ \color{red}{4} \\ 3 \end{matrix}\right|\begin{matrix} \color{red}{2} \\ 5 \\ \end{matrix}\right| + \left|\begin{matrix} 1 \\ 2 \\ \color{red}{4} \end{matrix}\right|\begin{matrix} \color{red}{3} \\ 5 \\ \end{matrix}\right|$$

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A New Relation

Definition (B.,Friedmann)

For λ of shape $2^m 1^{n-m}$ with $n \geq m$,

$$\eta([\mathbf{t}]) = m[\mathbf{t}] - \sum [s]$$

where the sum ranges over all tableaux s obtained from t by swapping an entry in the first column with an entry in the second column.

A New Relation

$$\eta : \begin{array}{|c|c|} 1 & 4 \\ 2 & 5 \\ 3 & \end{array}$$

$$\downarrow$$

$$2 \begin{array}{|c|c|} 1 & 4 \\ 2 & 5 \\ 3 & \end{array} - \left(\begin{array}{|c|c|} {\color{red}4} & {\color{red}1} \\ 2 & 5 \\ 3 & \end{array} + \begin{array}{|c|c|} 1 & {\color{red}2} \\ {\color{red}4} & 5 \\ 3 & \end{array} + \begin{array}{|c|c|} 1 & {\color{red}3} \\ 2 & 5 \\ {\color{red}4} & \end{array} + \begin{array}{|c|c|} {\color{red}5} & {\color{red}1} \\ 2 & 4 \\ 3 & \end{array} + \begin{array}{|c|c|} 1 & {\color{red}2} \\ {\color{red}5} & 4 \\ 3 & \end{array} + \begin{array}{|c|c|} 1 & {\color{red}3} \\ 2 & 4 \\ {\color{red}5} & \end{array} \right)$$

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Upshot: To generate $\ker(\alpha)$ for any partition shape, we only need to enumerate over pairs of adjacent columns in *column tabloids* with *ordered columns*.

Upshot 2: The Specht Module S^λ can be realized as a quotient of column tabloids by the η-relations!

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3. Compute the scalar action of η on each irreducible

(Brief) Motivation

Definition (Friedmann, 2011)

A *Lie algebra of the* n^{th} *kind (LAnKe)* is a vector space with an n-linear, anti-symmetric bracket and satisfying a generalized Jacobi Identity

$$[[x_1, \dots, x_n], x_{n+1}, \dots, x_{2n-1}] =$$

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Friedmann, Hanlon, Stanley and Wachs initiated the study of the representations of the symmetric group on the multilinear component of the free LAnKe.

$[[1, 2], 3]$

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$[[1,2],3]$ $\updownarrow$ $\begin{vmatrix} 1 & 3 \\ 2 & \end{vmatrix}$

$[[1,2,3],4,5]$ $\updownarrow$ $\begin{vmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \end{vmatrix}$

$[[1,2,3,4],5,6,7]$ $\updownarrow$ $\begin{vmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & \end{vmatrix}$

$$[[1,2],3] \qquad [[1,2,3],4,5] \qquad [[1,2,3,4],5,6,7]$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$\begin{array}{|c|c|} 1 & 3 \\ 2 & \end{array} \qquad \begin{array}{|c|c|} 1 & 4 \\ 2 & 5 \\ 3 & \end{array} \qquad \begin{array}{|c|c|} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & \end{array}$$

It turns out that the η relations are equivalent to the generalized Jacobi Identity in this *n*-ary Lie algebra

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Upshot: We can use Fulton's α map to construct a map between the representations of the symmetric group on the free LAnKe and relevant Specht modules

The CataLAnKe Theorem

Friedmann, Hanlon, Stanley and Wachs prove that the representation of S_{2n-1} on the multi-linear component of the free LAnKe on $2n-1$ generators is isomorphic to $S^{2^{n-1}}$.

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Using the α map and η relations, we can give an alternate proof of this result via a direct isomorphism:

$$[[1,2,3],4,5] \mapsto \alpha\left(\begin{array}{|c|c|} 1 & 4 \\ 2 & 5 \\ 3 & \end{array}\right)$$

Thank you!

References:

[1] S. Brauner and T. Friedmann *One Garnir to Rule them all: on Specht Modules and the CataLAnKe Theorem*, arXiv preprint arXiv:1803.10918 (2018).

[2] T. Friedmann. *Orbifold Singularities, Lie Algebras of the Third Kind (LATKes) and Pure Yang-Mills Matter*, J. Math. Phys. **52** (2011).

[3] T. Friedmann, P. Hanlon, R. Stanley, M. Wachs. *On a generalization of Lie*(k)*: a CataLAnKe Theorem*, arXiv preprint arXiv:1710.00376, (2017).

[4] W. Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*, Vol. 35, Ch. 7.4. Cambridge University Press 1997.

[5] B. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Ch.2, Springer Ed. 2, 2001.

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