One Garnir to Rule Them All

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Representations of the Symmetric Group

Irreducible representations of $S_n$ correspond to partitions $\lambda$ of $n$
Each irreducible of $S_n$ of shape $\lambda$ can be realized as a *Specht Module* $S^\lambda$
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\end{array}$

2. Turn it into a row tabloid: $\{\mathbf{t}\} = \begin{array}{cc}
1 & 3 \\
2 & 4
\end{array} = \begin{array}{cc}
3 & 1 \\
4 & 2
\end{array}$
Classical Specht Module Construction

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3. Construct a row polytabloid:

$$e_t = \frac{1 \quad 3}{2 \quad 4} - \frac{2 \quad 3}{1 \quad 4} - \frac{1 \quad 4}{2 \quad 3} + \frac{2 \quad 4}{1 \quad 3}$$
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$S^\lambda$ has a basis of row polytabloids $e_t$ where the corresponding $t$ is a *Standard Young Tableau*. 
From our motivation, it would be more convenient to realize the Specht Module $S^\lambda$ as a *quotient* of elements which are naturally anti-symmetric.
Column Tabloids

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[t] = \begin{array}{cc}
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2 & 3 \\
\end{array} = \begin{array}{|c|c|}
2 & 4 \\
1 & 3 \\
\end{array}$$
A Dual Construction

Fulton defines a map

$$\alpha : \text{column tabloids of shape } \lambda \rightarrow \text{Specht module of shape } \lambda$$
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$$\alpha : \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \mapsto \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$$
What is \( \ker(\alpha) \)?
A Dual Straightening Algorithm

Definition (Fulton)

A dual Garnir relations is

$$\pi_{c,k}(t) := \sum [s]$$

where the sum is over column tabloids obtained by exchanging the top $k$ elements in the $(c + 1)^{st}$ column in all possible ways with $k$ elements in the $c^{th}$ column.
A Dual Straightening Algorithm

\[ \pi_{1,1} \left( \begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 \end{array} \right) = \begin{array}{c|c} 4 & 1 \\ \hline 2 & 5 \\ \hline 3 & 3 \end{array} + \begin{array}{c|c} 1 & 2 \\ \hline 4 & 5 \\ \hline 3 & 3 \end{array} + \begin{array}{c|c} 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 4 \end{array} \]
Theorem (Fulton)

The relations

\[ [t] - \pi_{c,k}(t) \]

over all \( t, c \) and \( k \) generate \( \ker(\alpha) \).

In characteristic 0, only the \( \pi_{1,1} \) relations are needed.

Upshot: The Specht Module \( S_\lambda \) can be realized as a quotient module of the space of column tabloids by dual Garnir relations.
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**Upshot:** The Specht Module $S^\lambda$ can be realized as a quotient module of the space of column tabloids by dual Garnir relations.
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\[
\pi_{1,1} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 5 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}
\]
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\]

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\pi_{1,1} \begin{pmatrix} 1 & 5 \\ 2 & 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 2 & 5 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 5 & 4 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 \end{pmatrix}
\]
A New Relation

**Definition (B., Friedmann)**

For \( \lambda \) of shape \( 2^m 1^{n-m} \) with \( n \geq m \),

\[
\eta([t]) = m[t] - \sum [s]
\]

where the sum ranges over all tableaux \( s \) obtained from \( t \) by swapping an entry in the first column with an entry in the second column.
A New Relation

\[ \eta : \begin{array}{c|c|c} 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \end{array} \]

\[ \downarrow \]

\[ 2 \begin{array}{c|c|c} 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \end{array} - \left( \begin{array}{c|c|c} 4 & 1 \\ \hline 2 & 5 \\ \hline 3 \\ \end{array} + \begin{array}{c|c|c} 1 & 2 \\ \hline 3 & 4 \\ \hline 4 \\ \end{array} + \begin{array}{c|c|c} 1 & 3 \\ \hline 2 & 5 \\ \hline 3 \\ \end{array} + \begin{array}{c|c|c} 5 & 1 \\ \hline 2 & 4 \\ \hline 3 \\ \end{array} + \begin{array}{c|c|c} 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \end{array} + \begin{array}{c|c|c} 1 & 3 \\ \hline 2 & 4 \\ \hline 5 \\ \end{array} \right) \]
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The $\eta$ relations generate $\ker(\alpha)$. 
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**Upshot:** To generate $\ker(\alpha)$ for any partition shape, we only need to enumerate over pairs of adjacent columns in *column tabloids* with *ordered columns.*
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**Upshot**: To generate $\ker(\alpha)$ for any partition shape, we only need to enumerate over pairs of adjacent columns in *column tabloids* with *ordered columns*. 

**Upshot 2**: The Specht Module $S^\lambda$ can be realized as a quotient of column tabloids by the $\eta$-relations!
Proof Outline

1. Look at the two-column case
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2. Use anti-symmetry to describe the space of column tabloids as an induced (multiplicity-free!) representation of $S_{n+m}$

\[ e^n(x) e^m(x) = m \sum_{i=0}^{n} s_{2i}^1 (x) - i (x) \]

by the Pieri Rule
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   by the Pieri Rule)

3. Compute the scalar action of $\eta$ on each irreducible
(Brief) Motivation

Definition (Friedmann, 2011)

A Lie algebra of the $n^{th}$ kind (LAnKe) is a vector space with an $n$-linear, anti-symmetric bracket and satisfying a generalized Jacobi Identity

$$[[x_1, \ldots, x_n], x_{n+1}, \ldots, x_{2n-1}] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, x_{n+1}, \ldots, x_{2n-1}], x_{i+1}, \ldots, x_n]$$
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Friedmann, Hanlon, Stanley and Wachs initiated the study of the representations of the symmetric group on the multilinear component of the free LAnKe.
It turns out that the $\eta$ relations are equivalent to the generalized Jacobi Identity in this $n$-ary Lie algebra.

Upshot:
We can use Fulton’s $\alpha$ map to construct a map between the representations of the symmetric group on the free LAnKe and relevant Specht modules.
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Friedmann, Hanlon, Stanley and Wachs prove that the representation of $S_{2n-1}$ on the multi-linear component of the free LAnKe on $2n - 1$ generators is isomorphic to $S^{2n-1}$. 
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Using the $\alpha$ map and $\eta$ relations, we can give an alternate proof of this result via a direct isomorphism:

$$[[1, 2, 3], 4, 5] \mapsto \alpha \left( \begin{array}{cc}
1 & 4 \\
2 & 5 \\
3 & 
\end{array} \right)$$
Thank you!

References:


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