Enumerating Linear Systems on Graphs, Dynkin Diagrams and Beyond

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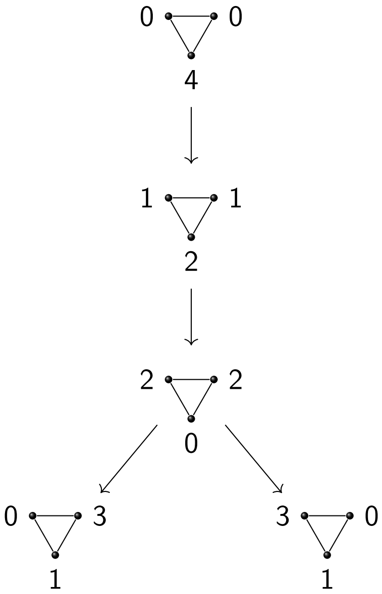
AMS Special Session on Divisors and Chip-Firing

Outline

1. Combinatorial question about effective divisors
2. Framework for answering this question
3. Answer via divisors
4. Answer via lattice points in polyhedra
5. Answer via Invariant Theory

A Combinatorial Question



The setup

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$$D = D(v_1)v_1 + \cdots + D(v_n)v_n$$

The degree of D is $\deg(D) = \sum_{v \in V} D(v)$

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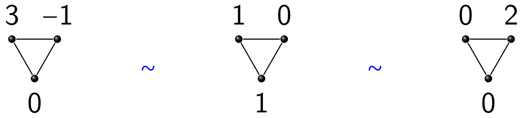
Linear equivalence: $D \sim D'$ if $D \underset{\textit{borrowing}}{\overset{\textit{lending}}{\rightsquigarrow}} D'$



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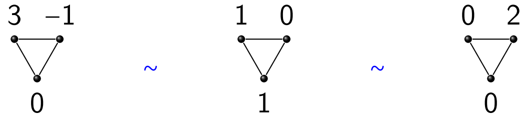

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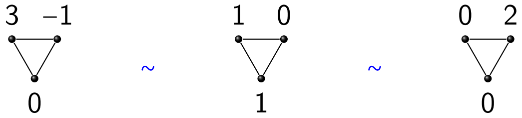

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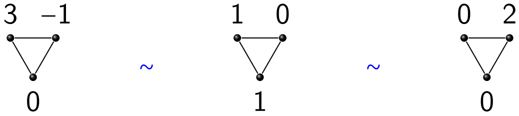

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L is singular with $\ker(L) = \mathbb{Z}\vec{1}$

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Complete linear system for $D \in \mathrm{Div}(G)$:

$$|D| = \{E \in \mathrm{Div}(G) : E \sim D \text{ and } E \text{ is effective}\}$$

$$= \text{all effective divisors linearly equivalent to } D$$

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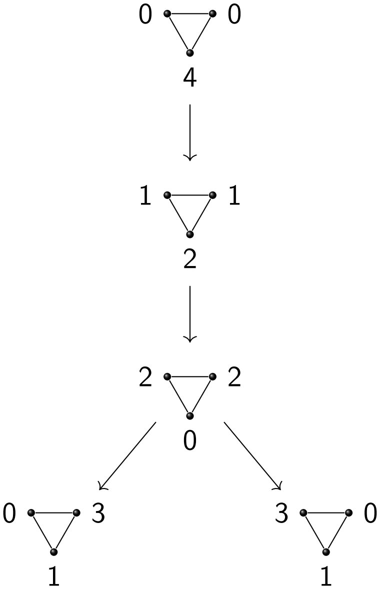
Goal: Enumerate $|D|$ for any graph G and divisor $D \in \mathrm{Div}(G)$.

A complete linear system



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Fix a vertex $\mathbf{q} \in V$. Removing the $\mathbf{q}^{th}$ row and column of L gives

$$\tilde{L} : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1},$$

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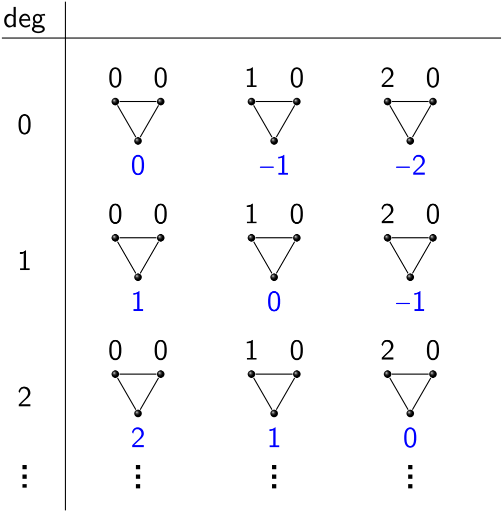
$$\mathrm{Jac}(G) = \mathrm{coker}(\tilde{L})$$

$$\begin{aligned} \mathrm{Pic}(G) &\overset{\sim}{\to} \mathrm{Jac}(G) \oplus \mathbb{Z} \\ [D] &\mapsto ([D - \deg(D)\mathbf{q}], \deg(D)). \end{aligned}$$

The strategy

Write $\mathrm{Pic}^{+}(G)$ by degree using $\mathrm{Jac}(G)$



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$$\text{Pic}^+(G) \qquad \begin{array}{c|ccccc}
\text{deg} & & & & & \\ \cline{1-5}
0 & [0] & [D_2] & \dots & [D_\kappa] & \leftarrow \ \text{Jac}(G) \\
1 & [q] & [D_2 + q] & \dots & [D_\kappa + q] & \\
2 & [2q] & [D_2 + 2q] & \dots & [D_\kappa + 2q] & \\
3 & [3q] & [D_2 + 3q] & \dots & [D_\kappa + 3q] & \\
\vdots & \vdots & \vdots & \vdots & \vdots &
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Underlying idea:

$$\begin{aligned} \mathrm{Pic}(G) &\xrightarrow{\sim} \mathrm{Jac}(G) \oplus \mathbb{Z} \\ [D] &\mapsto ([D - \deg(D)q], \deg(D)). \end{aligned}$$

The strategy

Use this setup to partition the set of effective divisors:

	deg					
$\mathrm{Pic}^{+}(G)$	0	$\lvert 0\rvert$	$\lvert D_2\rvert$	$\dots$	$\lvert D_\kappa\rvert$	$\leftarrow$ $\mathrm{Jac}(G)$
	1	$\lvert q\rvert$	$\lvert D_2 + q\rvert$	$\dots$	$\lvert D_\kappa + q\rvert$	
	2	$\lvert 2q\rvert$	$\lvert D_2 + 2q\rvert$	$\dots$	$\lvert D_\kappa + 2q\rvert$	
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Definition: For every $[D] \in \mathrm{Jac}(G)$, $\mathbb{E}_{[D]} := \bigcup_{k \geq 0} \lvert D + kq \rvert$

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Definition: For every $[D] \in \mathrm{Jac}(G)$, $\mathbb{E}_{[D]} := \bigcup_{k \geq 0} |D + kq|$

Goal: For every $[D] \in \mathrm{Jac}(G)$, compute $\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \#|D + kq| z^k$

Primary and secondary divisors

Theorem. (B, Glebe, Perkinson)

For every graph G there is a **unique** finite set

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secondary divisors: $\mathcal{S}_{[D]} \subset \mathbb{E}_{[D]}$

such that each $E \in \mathbb{E}_{[D]}$ can be written uniquely as

$$E = F + \sum_{P \in \mathcal{P}} a_P P$$

with $F \in \mathcal{S}_{[D]}$ and $a_P \in \mathbb{Z}_{\geq 0}$ for all $P \in \mathcal{P}$.

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Corollary.

$$\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \#|D + kq| z^k = \frac{\sum_{F \in \mathcal{S}_{[D]}} z^{\deg(F)}}{\prod_{P \in \mathcal{P}} (1 - z^{\deg(P)})}$$

Polyhedra

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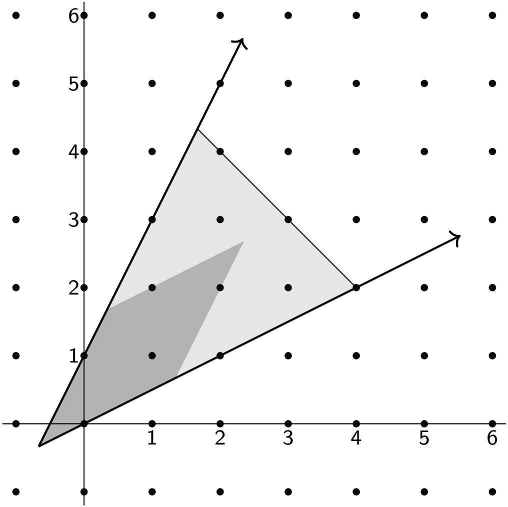


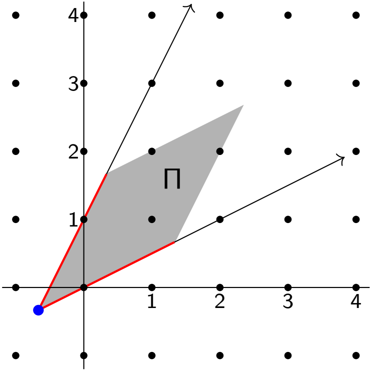
Polyhedra

A *rational simplicial pointed cone*

$$\mathcal{K} = \{p + \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_n\omega_n : \lambda_1, \ldots, \lambda_n \geq 0\}$$

generating rays = $\{\omega_1, \ldots, \omega_n\} \subset \mathbb{Z}^n$

fundamental parallelepiped $= \{\lambda_1, \ldots, \lambda_n : 1 > \lambda_1, \ldots, \lambda_n \geq 0\}$



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Turn this into a polyhedra:

$$\mathcal{K}_D := \{(f, t) \in \mathbb{R}^n \times \mathbb{R} : Lf + tq \geq -D \text{ and } f_n = 0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}.$$

Polyhedra

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$$\begin{aligned} \mathbb{E}_{[D]} &\longleftrightarrow \text{lattice points of } \mathcal{K}_D \\ \text{primary divisors } \mathcal{P} &\longleftrightarrow \text{generating rays } \{\omega_1, \dots, \omega_n\} \\ \text{secondary divisors } \mathcal{S}_{[D]} &\longleftrightarrow \text{ lattice points of fundamental parallelepiped} \end{aligned}$$

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Corollary.

Integer-point transform of $\mathcal{K}_D$ rediscovers

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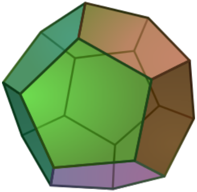
Invariant theory

$$\Phi_{\Gamma,\chi}(z) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{\overline{\chi(\gamma)}}{\det(I_n - z\gamma)}.$$



Invariant Theory

$$a_{(\lambda_1+n-1,\lambda_2+n-2,\dots,\lambda_n)}(x_1, x_2, \dots, x_n) = \det \begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \dots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \dots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \dots & x_n^{\lambda_n} \end{bmatrix}$$

Invariant theory

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Action of $\gamma \in \Gamma$ on $f \in \mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, \ldots, x_n]$ via matrix multiplication of indeterminates:

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$$\mathbb{C}[\mathbf{x}]^\Gamma := \{f \in \mathbb{C}[\mathbf{x}] : \gamma \cdot f = f \text{ for all } \gamma \in \Gamma\}.$$

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The χ-relative invariants for Γ are

$$\mathbb{C}[\mathbf{x}]^\Gamma_\chi := \{f \in \mathbb{C}[\mathbf{x}] : \gamma \cdot f = \chi(\gamma) f \text{ for all } \gamma \in \Gamma\}.$$

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Want generators for $\mathbb{C}[\mathbf{x}]^{\Gamma}_{\chi} := \{f \in \mathbb{C}[\mathbf{x}] : \gamma \cdot f = \chi(\gamma) f \text{ for all } \gamma \in \Gamma\}$

Invariant theory

Want generators for $\mathbb{C}[\mathbf{x}]_\chi^\Gamma := \{f \in \mathbb{C}[\mathbf{x}] : \gamma \cdot f = \chi(\gamma) f \text{ for all } \gamma \in \Gamma\}$

There exist algebraically independent *primary invariants*

$$p_1, \dots, p_n \in \mathbb{C}[\mathbf{x}]^\Gamma$$

and a list of χ*-relative invariants*:

$$q_1, \dots, q_t \in \mathbb{C}[\mathbf{x}]_\chi^\Gamma$$

such that

$$\mathbb{C}[\mathbf{x}]_\chi^\Gamma = \bigoplus_{i=1}^{t} q_i \mathbb{C}[p_1, \dots, p_n].$$

Invariant theory

Back to divisors:

For a fixed $q \in V$

$$\begin{tikzcd}
\mathbb{Z}^n = \mathrm{Div}(G) \arrow[r, two heads] & \mathrm{Pic}(G) \arrow[r, two heads] & \mathrm{Jac}(G) \\
D \arrow[r, mapsto] & {[D]} \arrow[r, mapsto] & {[D - \deg(D)q]}\,.
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Apply $\mathrm{Hom}(\,\cdot\,, \mathbb{C}^\times)$:

$$\mathrm{Jac}(G)^* \hookrightarrow \mathrm{Pic}(G)^* \hookrightarrow Div(G)^* \cong (\mathbb{C}^\times)^n \subset GL(\mathbb{C}^n),$$

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This induces a representation

$$\rho : \mathrm{Jac}(G)^* \longrightarrow \mathrm{GL}(\mathbb{C}^n)$$

Invariant Theory

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Character: For every $[D] \in \mathrm{Jac}(G)$

$$\begin{aligned} [D] : \Gamma &\longrightarrow \mathbb{C}^\times \\ \rho(\varphi) &\mapsto \varphi([D]) \end{aligned}$$

Invariant Theory

Theorem (B, Glebe, Perkinson)

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$$\text{secondary divisors } \mathcal{S}_{[D]} \longleftrightarrow \text{monomial [D]-relative invariants in } \mathbb{C}[\mathbf{x}]^{\Gamma}_{[D]}$$

Corollary.

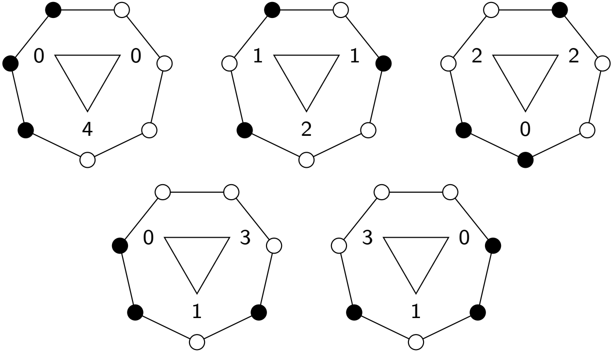
Molien's Theorem gives a new expression for $\Lambda_{[D]}(z)$:

$$\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \#|D + kq| z^k = \frac{1}{|\mathrm{Jac}(G)|} \sum_{\varphi \in \mathrm{Jac}(G)^*} \frac{\overline{\varphi([D])}}{\det(I_n - z\rho(\varphi))}$$

Necklaces

Theorem (B, Glebe, Perkinson) On the cyclic graph with n vertices,

$$\#|kq| = \text{number of binary necklaces with } n \text{ black beads and } k \text{ white beads.}$$



Chip-firing on M-matrices

The theory developed here holds in the broader context of chip-firing on certain types of M-matrices including:

Chip-firing on Dynkin Diagrams

Chip-firing on McCay-Cartan Matrices

Chip-firing on M-matrices

The theory developed here holds in the broader context of chip-firing on certain types of M-matrices including:

Chip-firing on Dynkin Diagrams

Chip-firing on McCay-Cartan Matrices

Need: An analogue of the Laplacian L and some technical conditions on its kernel so that there is an analog to

$$\mathrm{Pic}(G) \cong \mathrm{Jac}(G) \oplus \mathbb{Z}$$