Enumerating Linear Systems on Graphs, Dynkin Diagrams and Beyond

Sarah Brauner
(Joint with David Perkinson and Forrest Glebe)

University of Minnesota

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AMS Special Session on Divisors and Chip-Firing
Outline

1. Combinatorial question about effective divisors
2. Framework for answering this question
3. Answer via divisors
4. Answer via lattice points in polyhedra
5. Answer via Invariant Theory
A Combinatorial Question
The setup

\[ G = (V, E) \text{ is a connected graph} \]
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A divisor \( D \) on \( G \) is an element in \( \text{Div}(G) = \mathbb{Z}V \cong \mathbb{Z}^n \)
$G = (V, E)$ is a connected graph

A divisor $D$ on $G$ is an element in $\text{Div}(G) = \mathbb{Z}V \cong \mathbb{Z}^n$

$D \in \text{Div}(G)$ and $D(v_i) \in \mathbb{Z}$ can be written

$$D = D(v_1)v_1 + \cdots + D(v_n)v_n$$
The setup

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$D \in \text{Div}(G)$ and $D(v_i) \in \mathbb{Z}$ can be written

$$D = D(v_1)v_1 + \cdots + D(v_n)v_n$$

The degree of $D$ is $\text{deg}(D) = \sum_{v \in V} D(v)$
Linear equivalence: $D \sim D'$ if $\begin{vmatrix} 3 & -1 \\ 0 & 3 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \sim \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}$
The setup

Linear equivalence: $D \sim D'$ if $D \xrightarrow{lending} D'$ borrowing

\[
\begin{array}{ccc}
3 & -1 & 0 \\
\text{\includegraphics[width=0.2\textwidth]{triangle}} & \sim & \text{\includegraphics[width=0.2\textwidth]{triangle}} \\
0 & 1 & 0
\end{array}
\]

$L : \mathbb{Z}^n \to \mathbb{Z}^n$ is the (discrete) Laplacian
Linear equivalence: \( D \sim D' \) if \( D \xrightarrow{\text{lending}} \xrightarrow{\text{borrowing}} D' \)

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\( L : \mathbb{Z}^n \rightarrow \mathbb{Z}^n \) is the (discrete) Laplacian

\( \xrightarrow{\text{lending}} \xrightarrow{\text{borrowing}} \) is actually adding and subtracting columns of \( L \)
The setup

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\( L : \mathbb{Z}^n \to \mathbb{Z}^n \) is the \textit{(discrete) Laplacian}

\( \xrightarrow{\text{lending}} \xleftarrow{\text{borrowing}} \) is actually adding and subtracting columns of \( L \)

\( L \) is singular with \( \ker(L) = \mathbb{Z} \bar{1} \)
A divisor $D$ is effective if $D(\nu) \geq 0$ for all $\nu \in V$
A divisor $D$ is effective if $D(v) \geq 0$ for all $v \in V$

**Complete linear system** for $D \in \text{Div}(G)$:

$$|D| = \{ E \in \text{Div}(G) : E \sim D \text{ and } E \text{ is effective} \}$$

= all effective divisors linearly equivalent to $D$
A divisor $D$ is **effective** if $D(v) \geq 0$ for all $v \in V$

**Complete linear system** for $D \in \text{Div}(G)$:

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= all effective divisors linearly equivalent to $D$

**Goal:** Enumerate $|D|$ for any graph $G$ and divisor $D \in \text{Div}(G)$. 
A complete linear system
The setup

Pic(G) is group of divisors on G up to linear equivalence

Pic(G) = coker(L)
The setup

Pic\((G)\) is group of divisors on \(G\) up to linear equivalence

\[
\text{Pic}(G) = \text{coker}(L)
\]

Fix a vertex \(q \in V\). Removing the \(q^{th}\) row and column of \(L\) gives

\[
\tilde{L} : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1},
\]

the (non-singular!) reduced Laplacian
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Jac\((G)\) is group of degree 0 divisors on \(G\) up to linear equivalence

\[
\text{Jac}(G) = \text{coker}(\tilde{L})
\]
The setup

The setup

Pic($G$) is group of divisors on $G$ up to linear equivalence

$$\text{Pic}(G) = \text{coker}(L)$$

Fix a vertex $q \in V$. Removing the $q^{th}$ row and column of $L$ gives

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the (non-singular!) reduced Laplacian

Jac($G$) is group of degree 0 divisors on $G$ up to linear equivalence

$$\text{Jac}(G) = \text{coker}(\tilde{L})$$

$$\text{Pic}(G) \xrightarrow{\sim} \text{Jac}(G) \oplus \mathbb{Z}$$

$$[D] \mapsto ([D - \deg(D)q], \deg(D)).$$
The strategy

Write $\text{Pic}^+(G)$ by degree using $\text{Jac}(G)$

<table>
<thead>
<tr>
<th>deg</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>2</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>−1</td>
<td>−2</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
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</table>
For a fixed vertex $q \in V$
The strategy

For a fixed vertex \( q \in V \)

<table>
<thead>
<tr>
<th>( \text{deg} )</th>
<th>0</th>
<th>([0])</th>
<th>([D_2])</th>
<th>(\ldots)</th>
<th>([D_\kappa])</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>([q])</td>
<td>([D_2 + q])</td>
<td>(\ldots)</td>
<td>([D_\kappa + q])</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>([2q])</td>
<td>([D_2 + 2q])</td>
<td>(\ldots)</td>
<td>([D_\kappa + 2q])</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>([3q])</td>
<td>([D_2 + 3q])</td>
<td>(\ldots)</td>
<td>([D_\kappa + 3q])</td>
<td></td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
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</tbody>
</table>

\[ \text{Pic}^+(G) \quad \leadsto \quad \text{Jac}(G) \]

Underlying idea:
\[ \text{Pic}^+(G) \sim \text{Jac}(G) \oplus \mathbb{Z} \]

\[ \frac{D - \deg(D)}{q, \deg(D)} \]
The strategy

For a fixed vertex $q \in V$

<table>
<thead>
<tr>
<th>deg</th>
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<th>$[D_2]$</th>
<th>...</th>
<th>$[D_\kappa]$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>1</td>
<td>$[q]$</td>
<td>$[D_2 + q]$</td>
<td>...</td>
<td>$[D_\kappa + q]$</td>
</tr>
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Underlying idea:

Pic$(G) \sim \text{Jac}(G) \oplus \mathbb{Z}$

$[D] \mapsto ([D - \deg(D)q], \deg(D))$. 
The strategy

Use this setup to partition the set of effective divisors:

| deg | 0       | $|D_2|$ | ... | $|D_\kappa|$ |
|-----|---------|-------|-----|-------------|
| 0   | 0       |       |     |             |
| 1   | $|q|$   | $|D_2 + q|$  | ... | $|D_\kappa + q|$  |
| 2   | $2|q|$  | $|D_2 + 2q|$  | ... | $|D_\kappa + 2q|$  |
| 3   | $3|q|$  | $|D_2 + 3q|$  | ... | $|D_\kappa + 3q|$  |
| ... | ...     | ...   | ... | ...         |

$\text{Pic}^+(G) \leftarrow \text{Jac}(G)$
The strategy

Use this setup to partition the set of effective divisors:

| deg | 0 | $|D_2|$ | $|D_3|$ | $|D_κ|$ |
|-----|---|--------|--------|--------|
| 0   | 0 |        |        |        |
| 1   | $q$| $|D_2 + q|$ | $|D_κ + q|$ |        |
| 2   | $2q$| $|D_2 + 2q|$ | $|D_κ + 2q|$ |        |
| 3   | $3q$| $|D_2 + 3q|$ | $|D_κ + 3q|$ |        |
| ... |   |        |        |        |

$\textbf{Definition:}$ For every $[D] \in \text{Jac}(G)$, $\mathbb{E}_D := \bigcup_{k \geq 0} |D + kq|$
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Use this setup to partition the set of effective divisors:

| deg | \(0\) | \(\left| D_2 \right|\) | \(\ldots\) | \(\left| D_\kappa \right|\) |
|-----|------|----------------|--------|----------------|
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| 1   | \(\left| 2q \right|\) | \(\left| D_2 + 2q \right|\) | \(\ldots\) | \(\left| D_\kappa + 2q \right|\) |
| 2   | \(\left| 3q \right|\) | \(\left| D_2 + 3q \right|\) | \(\ldots\) | \(\left| D_\kappa + 3q \right|\) |
| \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) |

Definition: For every \([D] \in \text{Jac}(G)\), \(E[D] := \bigcup_{k \geq 0} \left| D + kq \right|\)

Goal: For every \([D] \in \text{Jac}(G)\), compute \(\Lambda[D](z) := \sum_{k=0}^{\infty} \#D + kq |z^k\)
Primary and secondary divisors

**Theorem.** (B, Glebe, Perkinson)

For every graph $G$ there is a **unique** finite set

primary divisors: $\mathcal{P} \subset \mathbb{F}_0$
Primary and secondary divisors

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and for every $[D] \in \text{Jac}(G)$, there is a **unique** finite set

**secondary divisors:** $\mathcal{S}_D \subset \mathbb{F}[D]$
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**secondary divisors:** $\mathcal{S}_{[D]} \subset \mathbb{F}_D$

such that each $E \in \mathbb{F}_D$ can be written uniquely as

$$E = F + \sum_{P \in \mathcal{P}} a_P P$$

with $F \in \mathcal{S}_{[D]}$ and $a_P \in \mathbb{Z}_{\geq 0}$ for all $P \in \mathcal{P}$. 
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**Corollary.**

$$\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \# |D + kq| z^k = \frac{\sum_{F \in \mathcal{S}_{[D]}} z^{\deg(F)}}{\prod_{P \in \mathcal{P}} (1 - z^{\deg(P)})}$$
A rational simplicial pointed cone

$$\mathcal{K} = \{ p + \lambda_1 \omega_1 + \lambda_2 \omega_2 + \cdots + \lambda_n \omega_n : \lambda_1, \ldots, \lambda_n \geq 0 \}$$

generating rays = $$\{ \omega_1, \ldots, \omega_n \} \subset \mathbb{Z}^n$$

fundamental parallelepiped = $$\{ \lambda_1, \ldots, \lambda_n : 1 > \lambda_1, \ldots, \lambda_n \geq 0 \}$$
Effective divisors are determined by a system of linear equations:
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\[ E \in \mathbb{E}_D \text{ if and only if there exists } f \in \mathbb{Z}^n \text{ and } t \in \mathbb{Z}_{\geq 0} \text{ such that } \]

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Turn this into a polyhedra:

\[ \mathcal{K}_D := \{(f, t) \in \mathbb{R}^n \times \mathbb{R} : Lf + tq \geq -D \text{ and } f_n = 0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}. \]
\[ \mathcal{K}_D := \{(f, t) \in \mathbb{R}^n \times \mathbb{R} : Lf + tq \geq -D \text{ and } f_n = 0\} \subset \mathbb{R}^n. \]

**Theorem.** (B, Glebe, Perkinson)

\( \mathcal{K}_D \) is a rational simplicial pointed cone and there are bijections
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**Theorem.** (B, Glebe, Perkinson)

\( K_D \) is a rational simplicial pointed cone and there are bijections

\[
\mathbb{E}_{[D]} \leftrightarrow \text{lattice points of } K_D
\]

primary divisors \( \mathcal{P} \leftrightarrow \) generating rays \( \{\omega_1, \ldots, \omega_n\} \)

secondary divisors \( \mathcal{S}_{[D]} \leftrightarrow \text{lattice points of fundamental parallelepiped} \)
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**Theorem.** (B, Glebe, Perkinson)

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\[ E[D] \longleftrightarrow \text{lattice points of } \mathcal{K}_D \]

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**Corollary.**

Integer-point transform of \( \mathcal{K}_D \) rediscovers

\[ \Lambda[D](z) := \sum_{k=0}^{\infty} \#|D + kq| z^k = \frac{\sum_{F \in \mathcal{S}_D} z^{\deg(F)}}{\prod_{P \in \mathcal{P}} (1 - z^{\deg(P)})} \]
Invariant theory

\[ \Phi_{\Gamma, \chi}(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi(\gamma)}{\det(I_n - z\gamma)}. \]
Invariant theory

\[ \Gamma \leq \text{GL}(\mathbb{C}^n) \text{ is a finite group} \]
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**Action** of \( \gamma \in \Gamma \) on \( f \in \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n] \) via matrix multiplication of indeterminates:

\[ \gamma \cdot f(x) := f(\gamma \cdot x). \]
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**Character** \(\chi : \Gamma \rightarrow \mathbb{C}^\times\)
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**Character** \( \chi : \Gamma \to \mathbb{C}^\times \)

The **invariant ring** for \( \Gamma \) is

\[
\mathbb{C}[x]^\Gamma := \{ f \in \mathbb{C}[x] : \gamma \cdot f = f \text{ for all } \gamma \in \Gamma \}.
\]
Invariant theory

$\Gamma \leq \operatorname{GL}(\mathbb{C}^n)$ is a finite group

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The invariant ring for $\Gamma$ is

$$\mathbb{C}[x]^\Gamma := \{ f \in \mathbb{C}[x] : \gamma \cdot f = f \text{ for all } \gamma \in \Gamma \}.$$

The $\chi$-relative invariants for $\Gamma$ are

$$\mathbb{C}[x]_\chi^\Gamma := \{ f \in \mathbb{C}[x] : \gamma \cdot f = \chi(\gamma)f \text{ for all } \gamma \in \Gamma \}.$$
Want generators for $\mathbb{C}[x]^\Gamma_\chi := \{ f \in \mathbb{C}[x] : \gamma \cdot f = \chi(\gamma)f \text{ for all } \gamma \in \Gamma \}$
Want generators for $\mathbb{C}[x]_{\chi}^\Gamma := \{ f \in \mathbb{C}[x] : \gamma \cdot f = \chi(\gamma)f \text{ for all } \gamma \in \Gamma \}$

There exist algebraically independent *primary invariants*

$$p_1, \ldots, p_n \in \mathbb{C}[x]_{\chi}^\Gamma$$

and a list of *$\chi$-relative invariants*:

$$q_1, \ldots, q_t \in \mathbb{C}[x]_{\chi}^\Gamma$$

such that

$$\mathbb{C}[x]_{\chi}^\Gamma = \bigoplus_{i=1}^t q_i \mathbb{C}[p_1, \ldots, p_n].$$
Back to divisors:

For a fixed $q \in V$

$$\mathbb{Z}^n = \text{Div}(G) \longrightarrow \text{Pic}(G) \longrightarrow \text{Jac}(G)$$

$$D \quad \mapsto \quad [D] \quad \mapsto \quad [D - \deg(D)q].$$
Invariant theory

Back to divisors:

For a fixed $q \in V$

$$\mathbb{Z}^n = \text{Div}(G) \longrightarrow \text{Pic}(G) \longrightarrow \text{Jac}(G)$$

$D \longmapsto [D] \longmapsto [D - \deg(D)q].$

Apply $\text{Hom}(\cdot, \mathbb{C}^\times)$:

$$\text{Jac}(G)^* \hookrightarrow \text{Pic}(G)^* \hookrightarrow \text{Div}(G)^* \cong (\mathbb{C}^\times)^n \subset GL(\mathbb{C}^n),$$
Back to divisors:

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Apply Hom($\cdot$, $\mathbb{C}^\times$):

$$\text{Jac}(G)^* \hookrightarrow \text{Pic}(G)^* \hookrightarrow \text{Div}(G)^* \cong (\mathbb{C}^\times)^n \subset GL(\mathbb{C}^n),$$

This induces a representation

$$\rho : \text{Jac}(G)^* \longrightarrow GL(\mathbb{C}^n)$$
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**Action:** $\Gamma := \rho(\text{Jac}(G)^*)$ acts on $\mathbb{C}[x]$
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**Action:** \( \Gamma := \rho(\text{Jac}(G)^*) \) acts on \( \mathbb{C}[x] \)

**Character:** For every \([D] \in \text{Jac}(G)\)

\[ [D] : \Gamma \rightarrow \mathbb{C}^\times \]

\[ \rho(\varphi) \mapsto \varphi([D]) \]
Invariant Theory

**Theorem** (B, Glebe, Perkinson)
For every $[D] \in \text{Jac}(G)$, there are bijections

\[
\begin{align*}
[D] & \leftrightarrow \text{monomial } C\text{-basis for } C[x] \\
\Gamma[D] & \leftrightarrow \text{monomial primary invariants in } C[x] \\
P[D] & \leftrightarrow \text{monomial } [D]\text{-relative invariants in } C[x] \\
S[D] & \leftrightarrow \text{secondary divisors}
\end{align*}
\]

**Corollary.** Molien's Theorem gives a new expression for $\Lambda[D](z)$:

\[
\Lambda[D](z) = \sum_{k=0}^{\infty} \frac{\sum_{\text{divides } \phi(D) + kq} \phi(D) \det(I_n - z\rho(\phi(D)))}{\text{Jac}(G)}
\]
Theorem (B, Glebe, Perkinson)
For every $[D] \in \text{Jac}(G)$, there are bijections

\[ E[D] \leftrightarrow \text{monomial } \mathbb{C}\text{-basis for } \mathbb{C}[x]^\Gamma_D \]

primary divisors $\mathcal{P} \leftrightarrow \text{monomial primary invariants in } \mathbb{C}[x]^\Gamma$

secondary divisors $\mathcal{S}[D] \leftrightarrow \text{monomial } [D]\text{-relative invariants in } \mathbb{C}[x]^\Gamma_D $
**Theorem** (B, Glebe, Perkinson)
For every \([D] \in \text{Jac}(G)\), there are bijections

\[
\mathbb{E}_{[D]} \leftrightarrow \text{monomial } \mathbb{C}\text{-basis for } \mathbb{C}[x]^{[D]}
\]

primary divisors \(\mathcal{P} \leftrightarrow \text{monomial primary invariants in } \mathbb{C}[x]^\Gamma\)

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**Corollary.**

Molien’s Theorem gives a new expression for \(\Lambda_{[D]}(z)\):

\[
\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \# |D + kq| z^k = \frac{1}{|\text{Jac}(G)|} \sum_{\varphi \in \text{Jac}(G)^*} \frac{\overline{\varphi([D])}}{\det(I_n - z \rho(\varphi))}
\]
**Theorem** (B, Glebe, Perkinson) On the cyclic graph with $n$ vertices,

$$\#|kq| = \text{number of binary necklaces with } n \text{ black beads and } k \text{ white beads.}$$
The theory developed here holds in the broader context of chip-firing on certain types of $M$-matrices including:

- Chip-firing on Dynkin Diagrams
- Chip-firing on McCay-Cartan Matrices
Chip-firing on $M$-matrices

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- Chip-firing on Dynkin Diagrams
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**Need**: An analogue of the Laplacian $L$ and some technical conditions on its kernel so that there is an analog to

$$\text{Pic}(G) \cong \text{Jac}(G) \oplus \mathbb{Z}$$