Enumerating Linear Systems on Graphs

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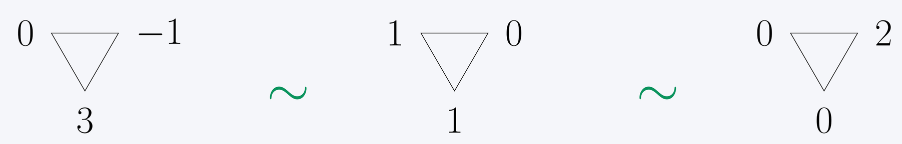
Chip-Firing on Graphs

The divisor theory of graphs views a finite connected graph $G = (V, E)$ as a discrete version of a Riemann surface. Fix $n = |V|$ and a sink vertex $q \in V$.

Terminology

A *divisor* D on G is an element of $\mathrm{Div}(G) := \mathbb{Z}V = \{\sum_{v\in V} D(v)v : D(v) \in \mathbb{Z}\}$, and the *degree* of a divisor D is $\deg(D) := \sum_{v\in V} D(v)$.

The *Laplacian of* G is the map $L : \mathbb{Z}^V \to \mathbb{Z}^V$ where L_{ii} is the valence of v_i and L_{ij} $(i \neq j)$ is $-\#\{$edges between v_i and $v_j\}$. We say D is linearly equivalent to E, written $D \sim E$, if there is a vector f such that $D + Lf = E$. For instance,



The *Jacobian (or critical) group* $\mathrm{Jac}(G)$ *of* G is the torsion part of $\mathrm{coker}(L)$.

Primary and Secondary Divisors

Theorem. For every graph G there is a finite set of primary divisors $\mathcal{P} \subset \mathbb{E}_{[0]}$ and for every $[D] \in \mathrm{Jac}(G)$, there is a finite set secondary divisors: $\mathcal{S}_{[D]} \subset \mathbb{E}_{[D]}$ such that each $E \in \mathbb{E}_{[D]}$ can be written uniquely as

$$E = F + \sum_{P\in\mathcal{P}} a_P P$$

with $F \in \mathcal{S}_{[D]}$ and $a_P \in \mathbb{Z}_{\geq 0}$ for all $P \in \mathcal{P}$.

Corollary.

$$\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \#|D + kq| z^k = \frac{\sum_{F\in\mathcal{S}_{[D]}} z^{\deg(F)}}{\prod_{P\in\mathcal{P}}(1 - z^{\deg(P)})}$$

Lattice Points in Polyhedra

Effective divisors are determined by a system of linear equations, which define a polytope

$$P_D := \{f \in \mathbb{R}^n \times : Lf \geq -D \text{ and } f_n = 0\} \subset \mathbb{R}^{n-1}.$$

Introducing another parameter for degree gives the polyhedron

$$\mathcal{K}_D := \{(f, t) \in \mathbb{R}^n \times \mathbb{R} : Lf + tq \geq -D \text{ and } f_n = 0\} \subset \mathbb{R}^n.$$

Theorem. $\mathcal{K}_D$ is a rational simplicial pointed cone and there are bijections

$\mathbb{E}_{[D]} \longleftrightarrow$ lattice points of $\mathcal{K}_D$

primary divisors $\mathcal{P} \longleftrightarrow$ integer generating rays pf $\mathcal{K}_D$

secondary divisors $\mathcal{S}_{[D]} \longleftrightarrow$ lattice points of fundamental parallelepiped of $\mathcal{K}_D$

Corollary. The integer-point transform of $\mathcal{K}_D$ rediscovers $\Lambda_{[D]}(z)$

Invariant Theory

A finite group $\Gamma \leq GL_n(\mathbb{C})$ acts on $\mathbb{C}[x_1, \ldots, x_n]$. For a character $\chi : \Gamma \to \mathbb{C}^\times$,

$$\mathbb{C}[x_1, \ldots, x_n]^\Gamma_\chi := \{f \in \mathbb{C}[x_1, \ldots, x_n] : \gamma \cdot f = \chi(\gamma) f \text{ for all } \gamma \in \Gamma\},$$

and is generated by finite sets of algebraically independent primary invariants in $\mathbb{C}[x_1, \ldots, x_n]^\Gamma$ and χ-relative invariants in $\mathbb{C}[x_1, \ldots, x_n]^\Gamma_\chi$.

How does this connect to divisors on graphs?

For a fixed $q \in V$, the projection $\mathbb{Z}^n \cong \mathrm{Div}(G) \longrightarrow \mathrm{Jac}(G)$ induces a map:

$$\rho : \mathrm{Jac}(G)^* \hookrightarrow \mathrm{Div}(G)^* \cong (\mathbb{C}^\times)^n \subset GL(\mathbb{C}^n).$$

$\Gamma := \rho(\mathrm{Jac}(G)^*)$ naturally acts on $\mathbb{C}[x_1, \ldots, x_n]$ by matrix multiplication

Every $[D] \in \mathrm{Jac}(G)$ can be realized as a character $[D] : \Gamma \to \mathbb{C}^\times$ by

$$[D] : \rho(\varphi) \mapsto \varphi([D]).$$

Theorem. For every $[D] \in \mathrm{Jac}(G)$, there are bijections

$\mathbb{E}_{[D]} \longleftrightarrow$ monomial $\mathbb{C}$-basis for $\mathbb{C}[\mathbf{x}]^\Gamma_{[D]}$

primary divisors $\mathcal{P} \longleftrightarrow$ monomial primary invariants in $\mathbb{C}[\mathbf{x}]^\Gamma$

secondary divisors $\mathcal{S}_{[D]} \longleftrightarrow$ monomial [D]-relative invariants in $\mathbb{C}[\mathbf{x}]^\Gamma_{[D]}$

Corollary. Molien's Theorem gives a new expression for $\Lambda_{[D]}(z)$:

$$\Lambda_{[D]}(z) := \sum_{k=0}^{\infty} \#|D + kq| z^k = \frac{1}{|\mathrm{Jac}(G)|} \sum_{\varphi\in\mathrm{Jac}(G)^*} \frac{\overline{\varphi([D])}}{\det(I_n - z\rho(\varphi))}.$$

Our Project

A divisor D is *effective* if $D(v) \geq 0$ for every $v \in V$. As in the case of Riemann surfaces, we are interested in the *complete linear system* of D:

$$|D| := \{E \in \mathrm{Div}(G) : E \text{ is effective and } E \sim D\}.$$

Question: For any divisor D on any graph G, what is the cardinality of $|D|$?

Approach: Effective divisors can be partitioned by $\mathrm{Jac}(G)$: for each $[D] \in \mathrm{Jac}(G)$,

$$\begin{aligned} \mathbb{E}_{[D]} : &= \cup_{k\geq 0} |D + kq| \\ &= \{E \in \mathrm{Div}(G) : E \text{ is effective and } E - \deg(E)q \sim D\}. \end{aligned}$$

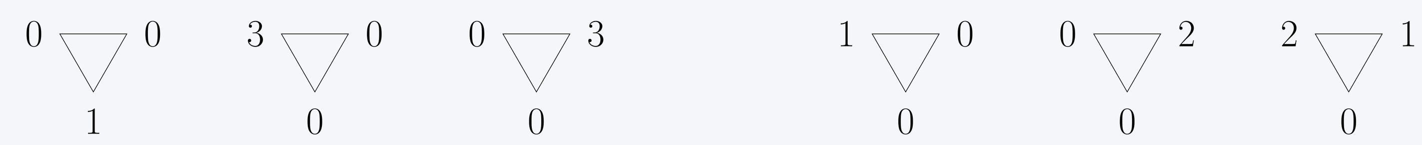
Goal: For each $[D] \in \mathrm{Jac}(G)$, compute generating functions

$$\Lambda_{[D]}(z) := \sum_{k\geq 0} \#|D + kq| z^k.$$

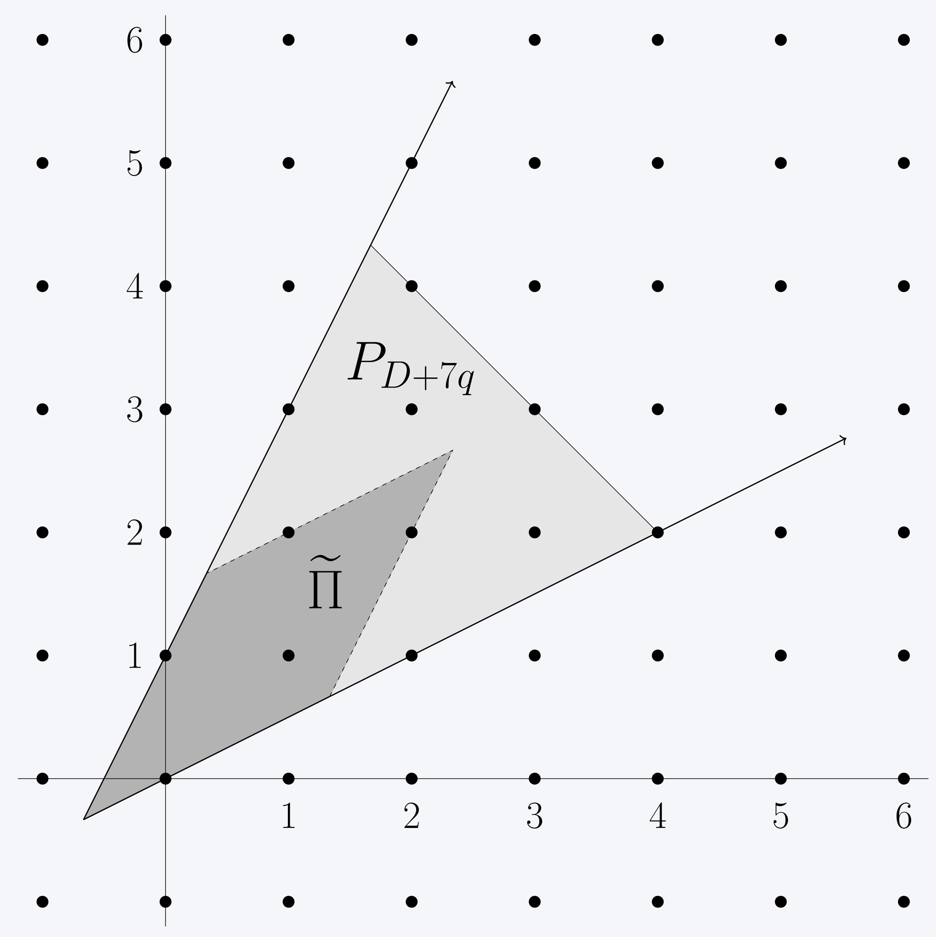
Example

Consider $G = C_3$, the cycle graph on 3 vertices (labeled clockwise), and $D = v_1 - v_3$.

Primary Divisors $\mathcal{P}$ for G: Secondary Divisors $\mathcal{S}_D$ for D:



We project $\mathcal{K}_D$ into $\mathbb{R}^2$ by its first two coordinates to get the cone $\widetilde{\mathcal{K}}_D$ shown below. Note that $\tilde{\Pi} \cap \mathbb{Z}^2$ bijects with $\mathcal{S}_{[D]}$ and the generating rays correspond to the second two primary divisors. The intersection of $\widetilde{\mathcal{K}}_D$ with the plane at height k has integer points in bijection with the elements of the complete linear system $|D + kq|$.

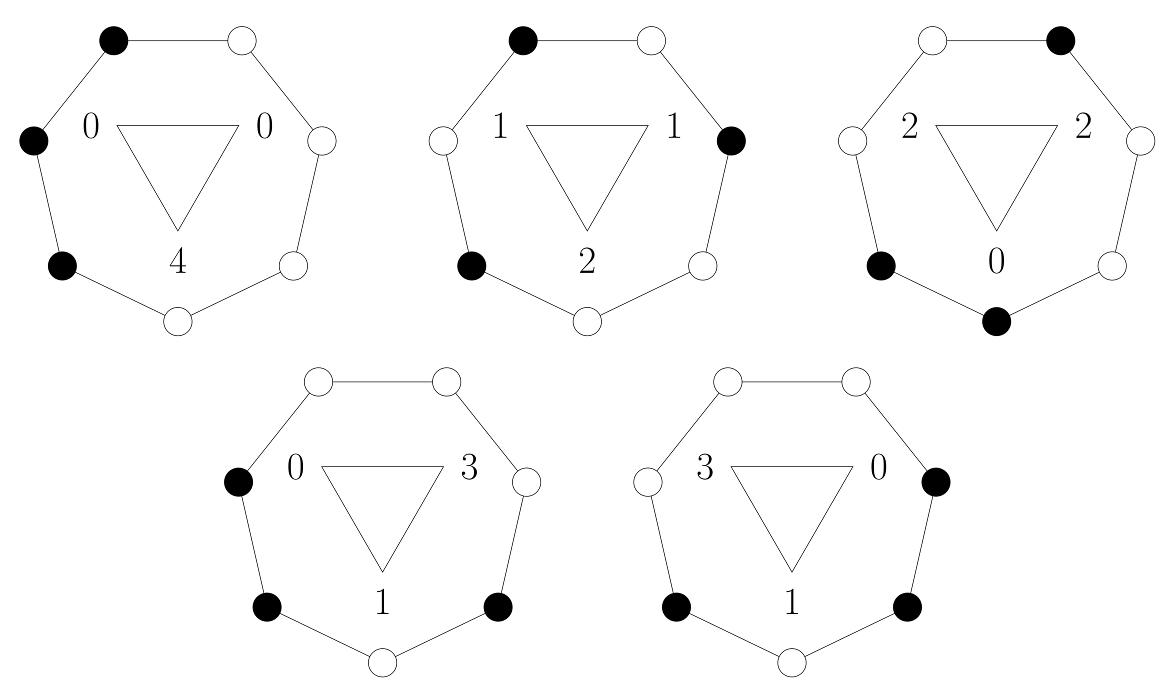


Connection to Necklaces

Theorem. On the cyclic graph with n vertices,

$\#|kq|$ = number of binary necklaces with n black beads and k white beads.

In the case that n and k are coprime, we have a combinatorial bijection, demonstrated below when $k = 4$ and $n = 3$:



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