EULERIAN REPRESENTATIONS FOR COINCIDENTIAL REFLECTION GROUPS

SARAH BRAUNER

UNIVERSITY OF MINNESOTA
braun622@umn.edu

BASED ON arXiv:2005.05953

UMN COMBINATORICS SEMINAR
OCTOBER 9, 2020
Big Idea:
Generalize a beautiful Type A story connecting combinatorics, representation theory and topology to a broader class of reflection groups

Outline:
1. Motivating Story: Type A
2. Coincidental reflection groups
3. Eulerian idempotents
4. The Varchenko-Gelfand ring
5. Main Results
Motivating Story: Type A
The story of the Eulerian idempotents begins with descents...

For $w = (w_1, \cdots, w_n) \in S_n$, the **descent set** of $w$ is

$$\text{Des}(w) := \{ i \in [n-1] : w_i > w_{i+1} \}$$

The **descent number** of $w$ is $\text{des}(w) := \# \text{Des}(w)$.

**Example:** If $w = (1, 4, 2, 5, 3)$, then $\text{Des}(w) = \{2, 4\}$ and $\text{des}(w) = 2$.

Equivalently, in the language of Coxeter groups:

$$\text{Des}(w) := \left\{ S_i \begin{array}{c} \text{transposition} \\ (i,i+1) \end{array} \in S : \ell(ws_i) < \ell(w) \right\},$$

**Remark:** $\text{Des}$ and $\text{des}$ can be defined for any Coxeter group.
Let $\mathbb{R} S_n$ be the group algebra of $S_n$ and $w = (w_1, w_2, \cdots w_n) \in S_n$.

**Surprising fact:** (Solomon, 1976)
There is a subalgebra of $\mathbb{R} S_n$ generated by sums of elements with the same descent set:

$$D(S_n) := \langle Y_T := \sum_{\substack{w \in S_n \\text{Des}(w) = T}} c_T w : c_T \in \mathbb{R}, T \subset [n - 1] \rangle$$

called **Solomon's descent algebra.**

**Example:** When $n = 3$, the descent algebra $D(S_3)$ has basis:

- $Y_{\emptyset} = (1, 2, 3)$
- $Y_1 = (2, 1, 3) + (3, 1, 2)$
- $Y_2 = (1, 3, 2) + (2, 3, 1)$
- $Y_{1,2} = (3, 2, 1)$.

**Remark:** In fact, $D(W)$ is a subalgebra for any Coxeter group.
Theorem (Garsia-Reutenauer, 1989).
There is a family of idempotents in $\mathbb{R}S_n$ defined by
\[
\sum_{k=0}^{n-1} t^{k+1} e_k = \sum_{w \in S_n} \left( t - 1 + n - \text{des}(w) \right)_n w.
\]
Call this family the **Eulerian idempotents**.

**Remark:**
By construction, the $e_k$ are in the Descent algebra $\mathcal{D}(S_n)$

In fact, the $e_k$ generate a commutative subalgebra of $\mathcal{D}(S_n)$ spanned by sums of elements with the same descent number

This subalgebra is known as the **Eulerian subalgebra**.
Example: When \( n = 3 \),

\[
e_0 = \frac{1}{6} \left( (1, 2, 3) - (2, 1, 3) - (3, 1, 2) - (1, 3, 2) - (2, 3, 1) + 2(3, 2, 1) \right)
\]

\[
= \frac{1}{6} \left( Y_\emptyset - Y_1 - Y_2 + 2Y_{1,2} \right)
\]

\[
e_1 = \frac{1}{2} \left( (1, 2, 3) - (3, 2, 1) \right)
\]

\[
= \frac{1}{2} \left( Y_\emptyset - Y_{1,2} \right)
\]

\[
e_2 = \frac{1}{6} \left( (1, 2, 3) + (2, 1, 3) + (3, 1, 2) + (1, 3, 2) + (2, 3, 1) + (3, 2, 1) \right)
\]

\[
= \frac{1}{6} \left( Y_\emptyset + Y_1 + Y_2 + Y_{1,2} \right)
\]
**Eulerian Ideempotents, Definition 2**

**Definition** (Barr, 1968).

The **Shuffle (Barr) element** in $\mathbb{R}S_n$ is

$$S := \sum_{i=1}^{n-1} \sum_{\substack{w \in S_n: \\
\text{Des}(w) \subseteq \{i\}}} w \in \mathcal{D}(S_n) \subset \mathbb{R}S_n.$$  

**Example:** When $n = 3$,

$$S = (1, 2, 3) + (2, 1, 3) + (3, 1, 2) + (1, 2, 3) + (1, 3, 2) + (2, 3, 1).$$

\[\text{Des}(w) \subseteq \{1\}\]

\[\text{Des}(w) \subseteq \{2\}\]

$$= 2(1, 2, 3) + (2, 1, 3) + (3, 1, 2) + (1, 3, 2) + (2, 3, 1).$$
Eulerian idempotents, Definition 2

Theorem (Gerstenhaber-Schack, 1987).

$S$ acts **semisimply** on $\mathbb{R}S_n$

$S$ has **eigenvalues** $\sigma_k := 2^{k+1} - 2$ for $0 \leq k \leq n - 1$.

Corollary.

By Lagrange interpolation, the **idempotent** projecting onto the $\sigma_k$-th eigenspace of $S$ is

$$e_k := \prod_{j \neq k} \frac{S - \sigma_j}{\sigma_k - \sigma_j}.$$

Theorem (Loday, 1989).

These idempotents are precisely the Eulerian idempotents.
**Eulerian Ideempotents for \( n = 3 \)**

**Example:** When \( n = 3 \), the Barr element \( S \) has eigenvalues 0, 2, 6:

\[
\varepsilon_0 = \frac{(S-2)(S-6)}{(0-2)(0-6)} \\
= \frac{1}{6} \left( (1, 2, 3) - (2, 1, 3) - (3, 1, 2) - (1, 3, 2) - (2, 3, 1) + 2(3, 2, 1) \right)
\]

\[
\sigma_0 = 0\text{-eigenspace projector}
\]

\[
\varepsilon_1 = \frac{(S-0)(S-6)}{(2-0)(2-6)} \\
= \frac{1}{2} \left( (1, 2, 3) - (3, 2, 1) \right)
\]

\[
\sigma_1 = 2\text{-eigenspace projector}
\]

\[
\varepsilon_2 = \frac{(S-0)(S-2)}{(6-0)(6-2)} \\
= \frac{1}{6} \left( (1, 2, 3) + (2, 1, 3) + (3, 1, 2) + (1, 3, 2) + (2, 3, 1) + (3, 2, 1) \right)
\]

\[
\sigma_2 = 6\text{-eigenspace projector}
\]
S\(_n\) acts on \(\mathbb{R} S\_n\) and \(\mathbb{R} S\_n e\_k\) by left multiplication...

The **Eulerian representations** are defined as the family of representations \(\mathbb{R} S\_n e\_k\) induced by this action.

By construction, \(\mathbb{R} S\_n e\_k\) is the \(\sigma\_k\)-eigenspace of \(S\).

**Example:** When \(n = 3\) for any \(\tau \in S\_3\),

\[
\tau \cdot \varepsilon_2 = \tau \cdot \frac{1}{6} \sum_{\sigma \in S\_3} \sigma = \frac{1}{6} \sum_{\sigma \in S\_3} \tau \sigma = \frac{1}{6} \sum_{\sigma' \in S\_3} \sigma' = \varepsilon_2.
\]
**Example: $n = 3$**

<table>
<thead>
<tr>
<th>$k$</th>
<th>Eulerian representation</th>
<th>Irreducible Decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\mathbb{R} S_3 \varepsilon_2 = \sigma_2$-eigenspace</td>
<td>![Diagram for $k=2$]</td>
</tr>
<tr>
<td>1</td>
<td>$\mathbb{R} S_3 \varepsilon_1 = \sigma_1$-eigenspace</td>
<td>![Diagram for $k=1$]</td>
</tr>
<tr>
<td>0</td>
<td>$\mathbb{R} S_3 \varepsilon_0 = \sigma_0$-eigenspace</td>
<td>![Diagram for $k=0$]</td>
</tr>
</tbody>
</table>
Question: Where do these representations naturally appear?

The answer is closely related to the braid arrangement,

\[ \mathcal{A}_n := \{ H_{ij} : 1 \leq i < j \leq n \} \]

where

\[ H_{ij} := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i = x_j\}. \]

**Example.** When \( n = 3 \), the essentialized braid arrangement \( \mathcal{A}_{S_3} \) is
Complement of the Braid Arrangement

The braid arrangement has \textbf{complement}

\[
\mathcal{M}(A_{S_n}) := \mathbb{R}^n \setminus A = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \neq x_j \text{ for } i, j \in [n] \right\}
\]

= the \( n \)-th ordered configuration space of \( \mathbb{R} \)

= \( \text{Conf}_n(\mathbb{R}) \).

We are interested in the \textit{d-thickened complement}

\[
\mathcal{M}^d(A_{S_n}) := \mathcal{M}(A) \otimes \mathbb{R}^d = \mathbb{R}^{dn} \setminus \left( \bigcup_{1 \leq i < j \leq n} H_{ij} \otimes \mathbb{R}^d \right)
\]

\[= \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j \text{ for } i, j \in [n] \right\}
\]

= the \( n \)-th ordered configuration space of \( \mathbb{R}^d \)

= \( \text{Conf}_n(\mathbb{R}^d) \).

\textbf{Example:} When \( d = 2 \), this is equivalent to the complement of the complexified arrangement \( \mathcal{M}(A) \otimes \mathbb{C} \)
A natural question:
what is $H^*(\text{Conf}_n(\mathbb{R}^d), \mathbb{R})$?


The ring $H^* \text{Conf}_n(\mathbb{R}^d)$ has presentation

$$\mathbb{R}\langle e_{ij} : 1 \leq i \neq j \leq n \rangle / J$$

where each $e_{ij}$ is in degree $d - 1$ and $J$ is generated by

1. $e_{ij}^2$
2. $e_{ij} = (-1)^d e_{ji}$
3. $e_{ij} e_{j\ell} + e_{j\ell} e_{i\ell} + e_{i\ell} e_{ij}$

for any $1 \leq i \neq j \neq \ell \leq n$.

This implies that $H^* \text{Conf}_n(\mathbb{R}^d)$ is

concentrated in degrees $k(d - 1)$ for $0 \leq k \leq n - 1$

**completely commutative** when $d$ is **odd**

**anti-commutative** when $d$ is **even**
The symmetric group $S_n$ acts on

$$\text{Conf}_n(\mathbb{R}^d) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{dn} : x_i \neq x_j \text{ for } i, j \in [n]\},$$

making $H^* \text{Conf}_n(\mathbb{R}^d)$ into an $S_n$-module...

**Known fact:** When $d$ is odd,

$$H^* \text{Conf}_n(\mathbb{R}^d) \cong_{S_n} \mathbb{R} \cdot S_n.$$

A more refined question:

What representation does $H^{k(d-1)} \text{Conf}_n(\mathbb{R}^d)$ carry for each $k$?

**Example:** $H^0 \text{Conf}_n(\mathbb{R}^d)$ is always the trivial representation.
Key connection:

When $d \geq 3$ is odd, for $0 \leq k \leq n - 1$,

$$H^{k(d-1)} \text{Conf}_n(\mathbb{R}^d) \cong S_n \mathbb{R} Sne_{n-1-k}.$$ 

How do we know?

1990 Hanlon computes the characters of $\mathbb{R} Sne_{n-1-k}$

1997 Sundaram-Welker prove an equivariant formulation of the Goresky-MacPherson formula relating cohomology of a subspace arrangement ←→ homology of its intersection lattice

As a special case:
they compute the characters of $H^{k(d-1)} \text{Conf}_n(\mathbb{R}^d)$
### Example: $n = 3$

**Example:** When $n = 3$ and $d$ is odd, 

<table>
<thead>
<tr>
<th>Eulerian representation</th>
<th>Configuration space cohomology</th>
<th>Irreducible decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R} S_3 e_2 = \sigma_2$-eigenspace</td>
<td>$H^0 \text{Conf}_3(\mathbb{R}^d)$ $= \mathbb{R}{1}$</td>
<td>$\square$</td>
</tr>
<tr>
<td>$\mathbb{R} S_3 e_1 = \sigma_1$-eigenspace</td>
<td>$H^{1(d-1)} \text{Conf}<em>3(\mathbb{R}^d)$ $= \mathbb{R}{e</em>{12}, e_{23}, e_{13}} / \mathcal{J}_1$</td>
<td>$\square + \square$</td>
</tr>
<tr>
<td>$\mathbb{R} S_3 e_0 = \sigma_0$-eigenspace</td>
<td>$H^{2(d-1)} \text{Conf}<em>3(\mathbb{R}^d)$ $= \mathbb{R}{e</em>{12}e_{23}, e_{12}e_{13}} / \mathcal{J}_2$</td>
<td>$\square$</td>
</tr>
</tbody>
</table>
Summary: For $0 \leq k \leq n - 1$, the following are equivalent as $S_n$-representations:

1. $H^{k(d-1)} \text{Conf}_n(\mathbb{R}^d)$ for odd $d \geq 3$;
2. The $k$-th graded piece of Cohen’s algebra for odd $d \geq 3$:
   $$\mathbb{R}\langle e_{ij} : 1 \leq i < j \leq n \rangle / \mathcal{J}$$
3. The $\sigma_{n-1-k} = \{2^{n-k} - 2\}$-eigenspace of the Barr’s shuffle element $S \in \mathbb{R} S_n$;
4. The representation $\mathbb{R} S_n \varepsilon_{n-1-k}$, where $\varepsilon_{n-1-k}$ is defined by
   $$\sum_{k=0}^{n-1} t^{k+1} \varepsilon_k = \sum_{w \in S_n} \left( t - 1 + n - \text{des}(w) \right) w.$$

Goal:
Generalize this statement to coincidental reflection groups, i.e. reflection groups whose exponents form an arithmetic progression $1, 1 + g, 1 + 2g, 1 + 3g, \cdots$
Recall the rising factorial \((t)_k := (t)(t + 1) \ldots (t + k - 1)\) and let

\[
\beta_{W,k}(t) := \left( \frac{t+g-1}{g} - k \right)_k \cdot \left( \frac{t+1}{g} \right)_{r-k}.
\]

**Theorem** (B—, 2020).
Let \(W\) be a real coincidental reflection group of rank \(r\). For \(0 \leq k \leq r\), the following are equivalent as \(W\)-representations:

1. \(H^{k(d-1)} M^d(A_W)\) for odd \(d \geq 3\)
2. \(V^k(A_W)\), the \(k\)-th graded piece of the associated graded Varchenko-Gelfand ring
3. The \(\sigma_{r-k}\)-th eigenspace of the shuffle element \(S(W) \in \mathbb{R} W\)
4. The representation \(\mathbb{R} W \varepsilon_{r-k}\) where \(\varepsilon_{r-k}\) is defined by

\[
\sum_{k=0}^{r} t^k \varepsilon_k = \sum_{w \in W} \beta_{W, \text{des}(w)}(t) \cdot w.
\]
1 Motivating Story: Type A

2 Coincidental reflection groups

3 Eulerian idempotents

4 The Varchenko-Gelfand ring

5 Main Results
COINCIDENTAL REFLECTION GROUPS
Every Coxeter group has a **reflection arrangement** $A_W$ where reflections $s \in W \leftrightarrow$ hyperplanes $H_s \in A_W$.

**Example**: The symmetric group $S_3$ acts on

\[ A_{S_3} = \]

The transposition $(ij) \in S_n$ reflects over the hyperplane $H_{ij}$
Every Coxeter group $W$ of rank $r$ has a unique set of integers $e_1 = 1 \leq e_2 \leq \cdots \leq e_r$ called the **exponents** of $W$, which satisfy many **product formulas**:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$S_n = A_{n-1}$</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>exponents</td>
<td>$1, 2, \cdots, n - 1$</td>
<td>$e_1, e_2, \cdots, e_r$</td>
</tr>
<tr>
<td>$#W$</td>
<td>$n! = 2 \cdot 3 \cdots n$</td>
<td>$\prod_{i=1}^{r} (1 + e_i)$</td>
</tr>
<tr>
<td>$\sum_{w \in W} q^{\ell(w)}$</td>
<td>$[n]_q! = [2]_q \cdot [3]_q \cdots [n]_q$</td>
<td>$\prod_{i=1}^{r} \frac{q^{1+e_i}-1}{q-1}$</td>
</tr>
<tr>
<td>$\sum_{w \in W} q^{\dim(V^w)}$</td>
<td>$(q + 1)(q + 2) \cdots (q + n - 1)$</td>
<td>$\prod_{i=1}^{r} (q + e_i)$</td>
</tr>
<tr>
<td>$\sum_{X \in \mathcal{L}(A_W)} \mu(V, X) q^{\dim(X)}$</td>
<td>$(q - 1)(q - 2) \cdots (q - n + 1)$</td>
<td>$\prod_{i=1}^{r} (q - e_i)$</td>
</tr>
</tbody>
</table>
**Coincidental Reflection Groups**

$W$ has exponents $e_1, e_2, \ldots, e_r$.

**Definition**

A reflection group is **coincidental** if its exponents form an arithmetic progression:

$$1, 1 + g, 1 + 2g, \ldots, 1 + (r - 1)g.$$  

for some integer $g$.

The *real* coincidental reflection groups are:

<table>
<thead>
<tr>
<th>$W$</th>
<th>$r$ := rank</th>
<th>exponents</th>
<th>$g$ := progression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_n$</td>
<td>$n - 1$</td>
<td>$1, 2, 3, \ldots, n - 1$</td>
<td>1</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$n$</td>
<td>$1, 3, 5, \ldots, 2n - 1$</td>
<td>2</td>
</tr>
<tr>
<td>$H_3$</td>
<td>3</td>
<td>$1, 5, 9$</td>
<td>4</td>
</tr>
<tr>
<td>$I_2(m)$</td>
<td>2</td>
<td>$1, m - 1$</td>
<td>$m - 2$</td>
</tr>
</tbody>
</table>
Outline

1 Motivating Story: Type A
2 Coincidental reflection groups
3 Eulerian idempotents
4 The Varchenko-Gelfand ring
5 Main Results
EULERIAN IDEMPOTENTS
Recall that $e_k \in \mathbb{R} S_n$ were defined in two ways:

1. As the **idempotent projectors** onto the eigenspaces of the shuffle element $S$, and
2. Via the **generating function**

$$\sum_{k=0}^{n-1} t^{k+1} e_k = \sum_{w \in S_n} \left( t - 1 + n - \text{des}(w) \right)^{n} w.$$

The Eulerian idempotents have been extensively studied and generalized since then!
1992: Bergeron-Bergeron define a Type $B$ analog:

$$
\sum_{k=0}^{n} t^k \epsilon_k = \sum_{w \in B_n} \left( \frac{t^{n-1}}{2^n} + n - \text{des}(w) \right)^w.
$$

1992: Bergeron-Bergeron-Howlett-Taylor define a finer family of idempotents in $\mathcal{D}(W)$ for any reflection group $W$

The idempotents are indexed by descent sets; summing over idempotents with the same descent size recovers the $\epsilon_k$

2009: Saliola constructs for any central arrangement $A$, a family of idempotents $\epsilon_X$ for each flat $X \in \mathcal{L}(A)$

In the case that $A$ is a reflection arrangement, the $\epsilon_X$ can be realized in $\mathbb{R} W$

2017: Aguiar-Mahajan further develop the theory of $\epsilon_X$, particularly for coincidental reflection groups
Upshot:
For any reflection group, these definitions all recover the same family of idempotents $\varepsilon_k \in \mathbb{R}^W$ for $0 \leq k \leq r$...

Call this family the **Eulerian idempotents**.
A generalized shuffle element

Recall how Barr’s shuffle element was defined:

\[
S := \sum_{i=1}^{n-1} \sum_{w \in S_n : \text{Des}(w) \subseteq \{i\}} w \in \mathcal{D}(S_n) \subset \mathbb{R} S_n.
\]

**Definition** (B—, 2020). For any reflection group \( W \) with generators \( s_1, \cdots, s_r \), the **shuffle element** \( S(W) \) is defined by

\[
S(W) := \sum_{i=1}^{r} \sum_{w \in W : \text{Des}(w) \subseteq \{s_i\}} w \in \mathcal{D}(W) \subset \mathbb{R} W.
\]

**Example**: In \( B_2 \) with Coxeter generators \( s \) and \( t \),

\[
S(B_2) = 1 + s + ts + sts + 1 + t + st + tst
\]

\[\text{Des}(w) \subseteq \{s\}\] \[\text{Des}(w) \subseteq \{t\}\]
A GENERALIZED SHUFFLE ELEMENT

**Proposition** (B—, 2020).

\[ S(W) \] acts semisimply on \( \mathbb{R} W \) for any reflection group \( W \).

When \( W \) is coincidental,

\[ S(W) \] has \( r + 1 \) distinct, non-negative, integer eigenvalues
\[ \sigma_0 < \sigma_1 < \cdots < \sigma_r \] and,

the projector onto the \( \sigma_k \)-th eigenspace of \( S(W) \) recovers the Eulerian idempotents.

This allows us to generalize the Eulerian subalgebra:

**Theorem** (B—, 2020).

There is an **Eulerian subalgebra** of \( D(W) \) generated by sums of elements with the same descent number

*if and only if* \( W \) is coincidental.

This subalgebra is always commutative.
1  Motivating Story: Type A
2  Coincidental reflection groups
3  Eulerian idempotents
4  The Varchenko-Gelfand ring
5  Main Results
The Varchenko-Gelfand ring
Varchenko and Gelfand define **Heaviside functions** on $\mathcal{M}(\mathcal{A})$ by

$$e_{H_i}(v) = \begin{cases} 
1 & v \in H_i^+ \\
0 & v \in H_i^- 
\end{cases}$$

**Example:** In Type $A$, when $n = 3$:

Multiplication is point-wise:

$$e_{H_12} : H_{12} \quad e_{H_13} : H_{13} \quad e_{H_12} e_{H_13} : H_{12} e_{H_13} : H_{23}$$
**The Varchenko-Gelfand Ring**

**Definition/Theorem** (Varchenko-Gelfand, 1987).

The associated graded Varchenko-Gelfand ring $\mathcal{V}(\mathcal{A})$ has presentation

$$\mathbb{R}[e_{H_i} : H_i \in \mathcal{A}]/\mathcal{J}$$

where $\mathcal{J}$ is generated by:

1. **Idempotent relation**: $e_{H_i}^2$ for each $H_i \in \mathcal{A}$;
2. **Circuit relation**: For every circuit (e.g. minimal linear dependency) $C = (H_1, H_2, \cdots, H_m)$ in $\mathcal{A}$ such that $C = C^+ \sqcup C^-$,

$$\sum_{i=1}^{m} c(i)e_{H_1} \cdots \hat{e}_{H_i} \cdots e_{H_m}$$

where

$$c(i) = \begin{cases} 
1 & \text{if } H_i \in C^-, \\
-1 & \text{if } H_i \in C^+. 
\end{cases}$$
**Example**: In Type A, when $n = 3$:

There is one circuit: $C = \{H_{12}, H_{23}, H_{13}\}$, which can be partitioned uniquely into $C^+ = H_{12}, H_{23}$ and $C^- = H_{13}$ so that

$$H_{12}^+ \cap H_{23}^+ \cap H_{13}^- = \emptyset.$$ 

Hence

$$\mathcal{V}(\mathcal{A}_{S_3}) = \mathbb{R}[e_{H_{12}}, e_{H_{23}}, e_{H_{13}}]/\langle e^2_{H_{12}}, e^2_{H_{23}}, e^2_{H_{13}}, e_{H_{12}} e_{H_{23}} - e_{H_{12}} e_{H_{13}} - e_{H_{23}} e_{H_{13}} \rangle.$$ 

**Note**: This matches Cohen’s presentation of $H^* \text{Conf}_3(\mathbb{R}^d)$, $d$ odd.
Claim:

\( \mathcal{V}(\mathcal{A}_W) \) generalizes \( H^* \text{Conf}_n(\mathbb{R}^d) \) for odd \( d \geq 3 \)...

Recall:

\[
\text{Conf}_n(\mathbb{R}^d) = \mathbb{R}^{dn} \setminus \left( \bigcup_{1 \leq i < j \leq n} H_{ij} \otimes \mathbb{R}^d \right)
\]

Definition:

For any central hyperplane arrangement \( \mathcal{A} \) of rank \( r \),

\[
\mathcal{M}^d(\mathcal{A}) := \mathbb{R}^{rd} \setminus \left( \bigcup_{H_i \in \mathcal{A}} H_i \otimes \mathbb{R}^d \right)
\]

As in Type A, consider \( H^* \mathcal{M}^d(\mathcal{A}) \).
The cohomology of $\mathcal{M}^d(\mathcal{A})$ depends on the parity of $d$!

**Even case:**

**Theorem** (Orlik-Solomon, 1980). When $d \geq 2$ is **even**, there is a $W$-equivariant ring isomorphism

$$H^* \mathcal{M}^d(\mathcal{A}_W) \cong_W OS(\mathcal{A}_W),$$

where $OS(\mathcal{A}_W)$ is the Orlik-Solomon algebra of $\mathcal{A}_W$.

**Odd case:**

**Theorem** (Moseley, 2017). When $d \geq 3$ is **odd**, there is a $W$-equivariant ring isomorphism

$$H^* \mathcal{M}^d(\mathcal{A}_W) \cong_W \mathcal{V}(\mathcal{A}_W),$$

where $\mathcal{V}(\mathcal{A}_W)$ is the associated-graded Varchenko-Gelfand ring of $\mathcal{A}_W$. 

### A Comparison of the Odd and Even Cases

<table>
<thead>
<tr>
<th>$H^*\mathcal{M}^d(\mathcal{A}_W)$ is isomorphic to...</th>
<th><strong>even</strong> $d \geq 2$</th>
<th><strong>odd</strong> $d \geq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>multiplication in $H^*\mathcal{M}^d(\mathcal{A}_W)$</td>
<td>$\mathcal{OS}(\mathcal{A}_W)$</td>
<td>$\mathcal{V}(\mathcal{A}_W)$</td>
</tr>
<tr>
<td>presentation of $H^*\mathcal{M}^d(\mathcal{A}_W)$</td>
<td>anti-commutative</td>
<td>commutative</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{R}\langle e_H : H \in \mathcal{A} \rangle$</td>
<td>$\mathbb{R}[e_H : H \in \mathcal{A}]$</td>
</tr>
<tr>
<td></td>
<td>idempotent &amp; circuit relations</td>
<td>idempotent &amp; circuit relations</td>
</tr>
</tbody>
</table>

### Special Cases

<table>
<thead>
<tr>
<th>Special cases</th>
<th><strong>even</strong> $d \geq 2$</th>
<th><strong>odd</strong> $d \geq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cohen’s presentation of $H^* \text{Conf}_n(\mathbb{R}^d)$</td>
<td>$\mathcal{OS}(\mathcal{A}_{S_n})$</td>
<td>$\mathcal{V}(\mathcal{A}_{S_n})$</td>
</tr>
<tr>
<td>Xicotencatl’s presentation of $H^* \text{Conf}_n(\mathbb{Z}^2)^d$</td>
<td>$\mathcal{OS}(\mathcal{A}_{B_n})$</td>
<td>$\mathcal{V}(\mathcal{A}_{B_n})$</td>
</tr>
</tbody>
</table>
1. Motivating Story: Type A
2. Coincidental reflection groups
3. Eulerian idempotents
4. The Varchenko-Gelfand ring
5. Main Results
Main Results
Let $\beta_{W,k}(t) := \left( \frac{t+g-1}{g} - k \right)_k \cdot \left( \frac{t+1}{g} \right)_{r-k}. $

**Theorem (B—, 2020).**

Let $W$ be a real coincidental reflection group of rank $r$. For $0 \leq k \leq r$, the following are equivalent as $W$-representations:

1. $H^{k(d-1)}M^d(A_W)$ for odd $d \geq 3$,
2. $\mathcal{V}^k(A_W)$, the $k$-th graded piece of the associated graded Varchenko-Gelfand ring
3. The $\sigma_{r-k}$-th eigenspace of the shuffle element $S(W) \in \mathbb{R}W$
4. The representation $\mathbb{R}W\epsilon_{r-k}$ where $\epsilon_{r-k}$ is defined by

$$
\sum_{k=0}^{r} t^k \epsilon_k = \sum_{w \in W} \beta_{W,\text{des}(w)}(t) \cdot w.
$$
Thank you for listening!
Complex Reflection Groups:

There are complex (non-real) coincidental reflection groups. These are precisely Shephard groups, which are the symmetry groups of complex polytopes.

**Question:** To what extent does the story of the real Eulerian representations generalize to Shephard groups?

*I would love to discuss any ideas in this direction!*

**Properties of the Eulerian representations**

Many representation theoretic properties of $\varepsilon_k$ in Type A are not known in other types!

**Currently:** $\mathbb{R} S_n \varepsilon_k$ has a “hidden” $S_{n+1}$ action. I am working on generalizing this to type B using configuration spaces.
For $X \in \mathcal{L}(A)$, the restriction arrangement $A^X$ is

$$A^X := \{H \cap X : H \in A, X \not\subset H\}.$$ 

**Theorem:** (Abramenko, 1994; Aguiar-Mahajan, 2017).

$A^X$ is a reflection arrangement for every $X \in \mathcal{L}(A)$ if and only if $W$ is a (product of) coincidental reflection group(s).

When $W$ is coincidental: $A^X \simeq A^Y$ if and only if $\dim(X) = \dim(Y)$
Let $[X] \in \mathcal{L}(\mathcal{A})/W$ be the $W$-orbit of $X \in \mathcal{L}(\mathcal{A})$.

**Theorem** (B–, 2020).

For any finite Coxeter group $W$ and $[X] \in \mathcal{L}(\mathcal{A})/W$,

$$\mathbb{R} \left\{ We_{[X]} \right\} \cong_{W} \left\{ V(\mathcal{A})[X] \right\}$$

idempotent indexed by flat orbits  

decomposition of $V(\mathcal{A})$ by flat orbit
Big idea:
Map $S(W)$ into the Tits (face) semigroup algebra of $A$

Relate eigenvalues of $S(W)$ to restriction arrangements $A^X$

Use the fact that when $W$ is coincidental, $A^X$ depends only on the dimension of $X$