Practice Problems from Thursday

1. Is \( \sum_{n=1}^{\infty} \frac{1}{n^{n+1}} + \frac{1}{n^{n^2}} \) convergent?

   No. We have \( \sum_{n=1}^{\infty} \frac{1}{n^{n+1}} + \frac{1}{n^{n^2}} \geq \sum_{n=1}^{\infty} \frac{1}{n^{n+1}} \) 

   and \( \lim_{n \to \infty} \frac{1}{n^{n+1}}, \frac{1}{n^{n^2}} = 1 \). The series \( \sum_{n=1}^{\infty} \frac{1}{n^n} \) diverges by the p-series test, so \( \sum_{n=1}^{\infty} \frac{1}{n^{n+1}} + \frac{1}{n^{n^2}} \) diverges by the comparison test.

   [Note: You need to use limit comparison and not comparison to show that \( \sum_{n=1}^{\infty} \frac{1}{n^{n+1}} \) diverges]

2. What is the Taylor series of \( f(x) = x^3 \cos x \) centered at \( x=0 \)?

   We know the Taylor series of \( \cos x \) at \( x=0 \) (the Maclaurin series) is \( \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \), so \( x^3 \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n)!} \).

3. Where does \( \sum_{n=2}^{\infty} \frac{2^n(x-3)^n}{n} \) converge?

   We use the ratio test:
   \[
   \lim_{n \to \infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{n+1} \cdot \frac{n}{2^n(x-3)^n} \right| = \lim_{n \to \infty} \left| 2(x-3) \right| \frac{n}{n+1} = \left| 2(x-3) \right|
   \]

   so converges absolutely for \( \left| 2(x-3) \right| < 1 \)
\[ |2(x-3)| < 1 \text{ means } -1 < 2(x-3) < 1 \]
\[-\frac{1}{2} < x-3 < \frac{1}{2} \]
\[ 5/2 < x < 7/2 \text{ (radius of convergence } = 1/2) \]

Now we check the endpoints:

At \( x = 5/2 \)
\[ \sum_{n=2}^{\infty} \frac{2^n (x-3)^n}{n} = \sum_{n=2}^{\infty} \frac{2^n (-\frac{1}{2})^n}{n} \]

this converges by the alternating series test, since \( \frac{1}{n} \) decreases to 0.

At \( x = 7/2 \)
\[ \sum_{n=2}^{\infty} \frac{2^n (x-3)^n}{n} = \sum_{n=2}^{\infty} \frac{2^n (\frac{1}{2})^n}{n} = \sum_{n=2}^{\infty} \frac{1}{n} \]

this diverges by the p-series test.

So
\[ \sum_{n=2}^{\infty} \frac{2^n (x-3)^n}{n} \text{ converges for on the interval } [5/2, 7/2) \]