1. A container in the shape of a paraboloid (a parabola revolved around an axis going through the vertex of the parabola and the directrix, so every horizontal slice, or cross section, is a circle.) is being filled with a liquid. The paraboloid is 2 feet in height and at its maximum width is 14 inches wide. Find: (Do **NOT** use information from one part in another part!)

(a) the rate at which the height is changing when the liquid is 10 inches high if the volume is changing at 0.125 ft³/s

(b) the rate at which the radius is changing when the liquid is at a radius of 6 inches if the volume is changing at 0.125 ft³/s

(c) the rate at which the volume is changing when the liquid is 12 inches high and at a radius of 7 inches if the radius is changing at 0.5 ft/s and the height is changing at 0.25 ft/s

\[ V_{\text{paraboloid}} = \frac{1}{2} \pi r^2 h \]

\[ \frac{r}{7/12} = \frac{h}{2} \quad \text{so} \quad r = \frac{7}{24} h \quad \text{or} \quad h = \frac{24}{7} r \]

\[ a) \quad \frac{dh}{dt} = ? \quad \frac{dV}{dt} = \frac{1}{8} \text{ ft}^3/\text{s} \quad h = 10 \text{ in.} \quad \frac{2}{7} \text{ ft} \]

\[ V = \frac{1}{2} \pi r^2 h = \frac{1}{2} \pi \left( \frac{7}{24} h \right)^2 h = \frac{49}{1152} \pi h^3 \]

\[ \frac{dV}{dt} = \frac{49}{384} \pi h^2 \frac{dh}{dt} \quad \text{so} \quad \frac{dh}{dt} = \frac{384}{49} \pi \frac{1}{h} \frac{dV}{dt} \]

\[ = \frac{384}{49} \pi \left( \frac{24}{7} \right) \left( \frac{2}{7} \right) \]

\[ \approx 0.0494 \text{ ft}^3/\text{s} \]

\[ b) \quad \frac{dr}{dt} = ? \quad \frac{dV}{dt} = \frac{1}{8} \text{ ft}^3/\text{s} \quad r = 6 \text{ in.} \quad \frac{1}{2} \text{ ft} \]

\[ V = \frac{1}{2} \pi r^2 h = \frac{1}{2} \pi \left( \frac{3}{2} r \right)^2 = \frac{123}{7} \pi r^3 \]

\[ \frac{dV}{dt} = \frac{3}{7} \pi r^2 \frac{dr}{dt} \quad \text{so} \quad \frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{7}{3 \pi} \cdot \frac{1}{r^2} \]

\[ = \frac{1}{16} \cdot \frac{7}{3 \pi} \cdot \left( \frac{24}{7} \right)^2 \]

\[ = 0.0309 \text{ ft/s} \]

\[ c) \quad \frac{dV}{dt} = ? \quad h = 12 \text{ in.} \quad r = 6 \text{ in.} \quad \frac{1}{2} \text{ ft} \]

\[ \frac{dr}{dt} = 0.5 \text{ in./s} \quad \frac{dV}{dt} = 0.25 \text{ ft}^3/\text{s} \]

\[ V = \frac{1}{2} \pi r^2 h \]

\[ \frac{dV}{dt} = \frac{1}{2} \pi \left[ r^2 \frac{dh}{dt} + h (\frac{dr}{dt}) \frac{dV}{dt} \right] \]

\[ = \frac{1}{2} \pi \left[ \frac{1}{4} \cdot \frac{1}{4} + 1 \cdot 0.5 \cdot \frac{1}{2} \right] \]

\[ = \frac{1}{2} \pi \left[ \frac{1}{16} + \frac{1}{4} \right] = \frac{9}{32} \text{ ft}^3/\text{s} \]
2. Consider the function \( f(x) = e^x \sin(x) \) for \(-\pi \leq x \leq \pi\).

State the intervals of increase and decrease. State and justify any extrema based on the intervals of increase and decrease only. (Hint: Use the First Derivative Test.)

\[
\frac{d^2f}{dx^2} = e^x \cos x + \sin x e^x = 0
\]
\[
e^x (\cos x + \sin x) = 0
\]
\[
\text{no critical points as always positive}
\]
\[
\Rightarrow \cos x + \sin x = 0
\]
\[
-\cos x = \sin x
\]
\[
-1 = \tan x
\]
\[
in \text{2nd and 4th quadrants} \Rightarrow x = \frac{-\pi}{4}, \frac{3\pi}{4}
\]

Critical points: \(\frac{-\pi}{4}, \frac{3\pi}{4}, -\pi, \pi\)

\[
f(x)
\]
\[
f'(x)
\]
\[
\frac{d}{dx} f(x)
\]
\[
-\pi - \frac{\pi}{4} + \frac{3\pi}{4} - \pi
\]

By 1st Derivative Test, there is a \(\underline{\text{min}}\) at \(x = \frac{-\pi}{4}\), and a \(\underline{\text{max}}\) at \(x = \frac{3\pi}{4}\).

\[
limit_{x \to -\pi^+} f'(x) < 0 \Rightarrow \text{By Left Endpt Thm, Max at } x = -\pi
\]

\[
limit_{x \to -\pi} f''(x) < 0 \Rightarrow \text{By Right Endpt Thm, Min at } x = \pi.
\]
3. Determine the intervals of concavity, points of inflection and classify (with justification) any extrema via the Second Derivative Test of the function

\[ f(x) = x^6 - 6x^4 \text{ for } -5 \leq x \leq 5 \]

\[ f'(x) = 6x^5 - 24x^3 = 6x^3(x^2 - 4) = 0 \Rightarrow x = 0, \pm 2 \Rightarrow \text{critical points are } 0, 2, \pm 2. \]

\[ f''(x) = 30x^4 - 72x^2 = 6x^2(5x^2 - 12) \Rightarrow 0, \pm \sqrt{\frac{12}{5}} \text{ are potential points of inflection.} \]

\[ f''(x) \begin{array}{c|c|c|c|c} \hline x & - & 0 & + & \text{max} & + & 0 & - \\
\hline \sqrt{\frac{12}{5}} & 0 & \sqrt{\frac{12}{5}} & & \end{array} \]

so \( \pm \sqrt{\frac{12}{5}} \) are points of inflection.

Also \( f(x) \) is \( \text{CC} \) on \((-\sqrt{\frac{12}{5}}, \sqrt{\frac{12}{5}})\) and \( \text{CCu} \) on \((-\infty, -\sqrt{\frac{12}{5}}) \cup (\sqrt{\frac{12}{5}}, \infty)\).

Also we see

\[ f''(2) > 0 \text{ so local min at } x = -2 \text{ by 2nd D. Test,} \]

\[ f''(2) < 0 \text{ so local min at } x = 2 \text{ by 2nd D. Test.} \]

\[ f''(0) = 0 \Rightarrow \text{2nd D. Test fails} \]

\[ \frac{f(x)}{f'(x)} \begin{array}{c|c|c|c|c} \hline x & - & 0 & + & \text{max} & + & 0 & - \\
\hline \end{array} \]

\[ \lim_{x \rightarrow -5^+} f'(x) < 0 \text{ so local max by Left Endpt Thm} \]

\[ \lim_{x \rightarrow 5^-} f'(x) > 0 \text{ so local max by Right Endpt Thm.} \]
4. Compute: (Note: \( n! = n(n-1)(n-2) \cdots (2)(1) \))

\[
\lim_{x \to 0} \frac{e^{5x} - 5x - 1}{\sin(x) - x} = \frac{\lim_{x \to 0} e^{5x} - \lim_{x \to 0} 5x - \lim_{x \to 0} 1}{\lim_{x \to 0} \sin(x) - \lim_{x \to 0} x} = \frac{\lim_{x \to 0} e^{5x} - \lim_{x \to 0} 5x - \lim_{x \to 0} 1}{\lim_{x \to 0} \sin(x) - \lim_{x \to 0} x} = \frac{1 - 0 - 1}{0 - 0} = \frac{0}{0}
\]

\[
\lim_{x \to 0} \frac{5e^{5x} - 5}{\cos(x) - 1} = \frac{5 \cdot 5 - 5}{1 - 1} = \frac{25 - 5}{0} = \frac{20}{0} = -\infty
\]

\[
\lim_{x \to \infty} \frac{x^5 e^{-x}}{5!} = \lim_{x \to \infty} \frac{x^5}{5! e^x} = \infty
\]

\[
\lim_{x \to \infty} \frac{5 \cdot 4}{5! \cdot e^x} = \infty
\]

\[
\lim_{x \to \infty} \frac{5 \cdot 4 \cdot 3 \cdot x^2}{5! \cdot e^x} = \infty
\]

\[
\lim_{x \to \infty} \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot x}{5! \cdot e^x} = \infty
\]

\[
\lim_{x \to \infty} \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5! \cdot e^x} = \lim_{x \to \infty} \frac{5^x}{5! \cdot e^x} = \lim_{x \to \infty} \frac{1}{5! \cdot e^x} = \frac{1}{\infty} = 0
\]
5. Bruno is on a budget and wants to build a storage shed with a square base for 1500 dollars. After taking a visit to Cheap Skates 'R Us, he finds out that for the walls it will cost him about 3 dollars per square foot, the floor will be 4 dollars per square foot and the ceiling will cost him 6 dollars per square foot. What dimensions of the shed would yield the greatest volume if there is also 100 dollars in miscellaneous expenses for the shed?

\[
\text{Cost} = 1500 = 12wh + 4w^2 + 6w^2 + 100
\]
\[
1400 = 12wh + 10w^2
\]
\[
700 = 6wh + 5w^2 \quad \Rightarrow \quad h = \frac{700 - 5w^2}{6w}
\]

\[
V = lwh = w^2h = w^2 \left( \frac{700 - 5w^2}{6w} \right) = \frac{700w - 5w^3}{6}
\]

\[
V' = \frac{700}{6} - \frac{15w^2}{6} = 0 \quad \Rightarrow \quad \frac{700}{15} = \frac{15}{6}w^2 \quad \Rightarrow \quad w^2 = \frac{700}{15} = \frac{140}{3}
\]

\[
w = \sqrt{\frac{140}{3}} \approx 6.83\text{ ft}
\]

\[
h = \frac{700 - \frac{700}{3}}{6\sqrt{\frac{140}{3}}} \approx 11.34\text{ ft}
\]

\[
V'' = -\frac{30w}{6} = -5w < 0 \quad \text{for all values of w}
\]

so any critical value w will be a local max by the 2nd D. Test.
6. Sketch the graph of the continuous function $f(x)$ and classify the extrema and points of inflection (with justification) if:

(a) $f'(-5) = 0$, $f'(-2) = 0$, $f'(0) = DNE$, $f'(3) = 0$, $f'(6) = DNE$

(b) $f''(-4) = 0$, $f''(0) = DNE$, $f''(1) = 0$, $f''(6) = DNE$

(c) $f'(x) < 0$ on $-2 < x < 0$, $x > 6$, $0 < x < 3$, and $x < -5$. $f'(x) > 0$ on $3 < x < 6$, $-5 < x < -2$

(d) $f''(x) > 0$ on $x < -4$, $1 < x < 6$, $x > 6$. $f''(x) < 0$ on $-4 < x < 0$, $0 < x < 1$

By 1st D. Test, Max at $x = -2, 6$; Min at $x = -5, 3$

$x = -4, 1, 6$ are points of inflection as it changes concavity here.
7. Prove via tangent lines that the following function is concave up at the point \((6, -18)\) if

\[ f(x) = x^2 - 10x + 6 \]

\[ f''(x) = 2x - 10 \]
\[ f''(6) = 2 \]

\[ y + 18 = 2(x - 6) \]
\[ y = 2x - 30 = T_6(x) \]

For CCU,

\[ f(x) > T_6(x) \]
\[ x^2 - 10x + 6 > 2x - 30 \]
\[ x^2 - 12x + 36 > 0 \]
\[ (x - 6)^2 > 0 \rightarrow \text{always true for all } x \neq 6 \]

so CCU at \((6, -18)\).