1. Convert the following definite integral into another integral with the variable $\theta$ using the substitution $8x = 9\sin \theta$. Simplify the resulting integral involving $\theta$. Do not evaluate the final integral involving $\theta$. No credit without work.

$$8x = 9\sin \theta \Rightarrow x = \frac{9}{8}\sin \theta$$

$$dx = \frac{9}{8}\cos \theta \, d\theta$$

$$\int_{9/16}^{3/2} \left( \frac{x^2 \, dx}{\sqrt{81 - 64x^2}} \right) = \frac{1}{9} \int_{\frac{9\sin \theta}{8}}^{\frac{1}{16}} \frac{x^2 \, dx}{\sqrt{1 - \left( \frac{8}{9}x \right)^2}}$$

$$= \frac{1}{9} \int_{\frac{9\sin \theta}{8}}^{\frac{1}{16}} \left( \frac{9}{8} \sin \theta \right)^2 \, d\theta = \frac{9}{8} \int_{\frac{9\sin \theta}{8}}^{\frac{1}{16}} \sin^2 \theta \, d\theta$$

$$= \frac{81}{512} \int_{\frac{9\sin \theta}{8}}^{\frac{1}{16}} \sin^2 \theta \, d\theta$$

2. Evaluate the following integral using a $u$ substitution. Show all steps. Answers without work get zero credit.

$$u = \sqrt{7x + 16}$$

$$\int (x\sqrt{7x + 16}) \, dx$$

Let $u^2 = 7x + 16$, then $2udu = 7dx$.

Or $x = \frac{u^2 - 16}{7}$

$$= \int \frac{u^2 - 16}{7} \cdot u \cdot \frac{2}{7} \, du$$

$$= \frac{2}{49} \int (u^4 - 16u^2) \, du$$

$$= \frac{2}{49} \left[ \frac{u^5}{5} - \frac{16}{3} u^3 \right] + C$$

$$= \frac{2}{49} \left[ \frac{(7x + 16)^{5/2}}{5} - \frac{16}{3} (7x + 16)^{3/2} \right] + C$$
3. Find the general solution of the following differential equation. Solve for y in the solution.

\[ \frac{dy}{dt} = (y-6)(y+5) \]

\[ \int \frac{dy}{y-6(y+5)} = \int dt \]

\[ \Rightarrow A \cdot \frac{1}{y-6} + B \cdot \frac{1}{y+5} = \frac{1}{y-6(y+5)} \]

\[ B(y-6) + A(y+5) = 1 \]

\[ A + B = 0 \Rightarrow A = -B \]

\[ -6B + 5A = 1 \]

\[ -11B = 1 \]

\[ B = \frac{-1}{11} \]

\[ A = \frac{1}{11} \]

\[ \Rightarrow \ln \left( \frac{y-6}{y+5} \right) = 11t + C \]

\[ \frac{y-6}{y+5} = Ce^{11t} \]

\[ y-6 = yCe^{11t} + 5Ce^{11t} \]

\[ y = yCe^{11t} + 5Ce^{11t} + 6 \]

\[ y = \frac{5Ce^{11t} + 6}{1 - Ce^{11t}} \]

4. Suppose \( f(x) \) is such that \( |f''(x)| \leq 2 \) for \( 1 \leq x \leq 7 \). Let \( E_T \) denote the error made when the integral \( \int_1^7 f(x) \, dx \) is approximated using the trapezoid rule. Find the smallest value of \( n \) such that \( |E_T| < 10^{-5} \).

Recall \( |E_T| \leq \frac{K(b-a)^3}{12n^2} \).

\[ K = 2 \]

\[ b = 7 \]

\[ a = 1 \]

\[ |E_T| = 1 \times 10^{-5} \]

\[ 1 \times 10^{-5} \geq \frac{2(6)^3}{12n^2} \]

\[ 1 \times 10^5 \leq \frac{12n^2}{432} \]

\[ \Rightarrow n \geq \frac{\sqrt{432 \times 1 \times 10^5}}{\sqrt{12}} \]

\[ n \geq 1897.36 \]

\[ \Rightarrow n = 1898 \]
5. Find the arc length of the curve \( y = \frac{(2x+3)^{3/2}}{2} \) for \( 3 \leq x \leq 8 \). Show all work.

\[
L = \int_{a}^{b} \sqrt{1 + (y')^2} \, dx
\]

\[
y' = \frac{3}{2} (2x+3)^{1/2}
\]

\[
(y')^2 = \frac{9}{4} (2x+3) = 18x+27
\]

\[
L = \int_{3}^{8} \sqrt{1 + 18x+27} \, dx
\]

\[
L = \int_{3}^{8} \sqrt{18x+27} \, dx - \frac{8}{3} \left[ \frac{(18x+27)^{3/2}}{3} \right]_{3}^{8}
\]

\[
= \frac{(2(27)^{3/2})}{27} - \frac{(82)^{3/2}}{27}
\]

6. Find the partial fraction decomposition of the following integral. You do not need to evaluate the integral. Show all work including any equations used to reach the decomposition.

\[
\int \frac{30x - 12}{(x^2 - 8)(x+1)} \, dx
\]

\[
\frac{30x - 12}{(x^2 - 8)(x+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 - 8}
\]

\[
30x - 12 = (A + B)x + C(x^2 - 8)
\]

\[
A + C = 0 \Rightarrow A = -C
\]

\[
A + B = 30 \Rightarrow B = 30 - A
\]

\[
B - 8C = -12 \Rightarrow (30 - A) + 8A = -12
\]

\[
30 + 7A = -12
\]

\[
7A = -42 \Rightarrow A = -6
\]

\[
A = -6 \Rightarrow C = 6 \Rightarrow B = 30 - A = 24
\]

\[
\int \frac{30x - 12}{(x^2 - 8)(x+1)} \, dx = \int \frac{-6x + 36}{x^2 - 8} \, dx + \int \frac{24}{x+1} \, dx
\]
7. Explain why each of the following integrals is improper. Evaluate each of them and state whether they are convergent or divergent. All steps must be shown.

\[
\int_0^8 \frac{dx}{(8-x)^{5/3}}
\]

improper b/c function is undefined at \( x = 8 \)

\[
= \lim_{t \to 8^-} \int_0^t \frac{dx}{(8-x)^{5/3}} = \lim_{t \to 8^-} \int_0^t \frac{du}{u^{5/3}} = \lim_{t \to 8^-} \left[ \frac{3}{2} u^{2/3} \right]_0^t = \lim_{t \to 8^-} \left[ \frac{3}{2} (8-t)^{2/3} \right] = \text{divergent}
\]

undefined

(b)

\[
\int_0^\infty \frac{dx}{(8+x)^{5/3}}
\]
cannot evaluate at \( \infty \)

\[
= \lim_{t \to \infty} \int_0^t \frac{du}{(8+u)^{5/3}} = \lim_{t \to \infty} \left[ \frac{3}{2} (8+u)^{2/3} \right]_0^t = \lim_{t \to \infty} \left[ \frac{3}{2} (8)^{2/3} - \frac{3}{2} (8+t)^{2/3} \right]
\]

\[
= \frac{3}{2} \cdot \frac{1}{8^{2/3}} = \frac{3}{8} \]

convergent
8. The region bounded by the parabola \( y = -(x - 3)^2 + 4 \) and the \( x \)-axis is covered by a lamina of constant density \( \rho \). Set up but do not evaluate the integrals for \( M_x \) and \( M_y \), the moments about the \( x \)-axis and \( y \)-axis.

\[
y = -(x - 3)^2 + 4 = -(x^2 - 6x + 9 + 4) = -x^2 + 6x - 5 = -(x^2 - 6x + 5) = -(x-5)(x-1)
\]

\[
M_x = \frac{\rho}{2} \int_{0}^{5} \left[ -(x^2 - 6x + 5) \right] \, dx
\]

\[
M_y = \rho \int_{0}^{5} (x)(-x^2 + 6x - 5) \, dx
\]
9. Solve the initial value problem by solving for $y$ in terms of $x$.

\[ \frac{dy}{dx} = \frac{4(1-y)}{x+5} \text{ and } y(1) = 2 \]

\[ \int \frac{dy}{1-y} = \int \frac{4}{x+5} \, dx \]

Let $u = 1 - y$.

\[ -\ln u = 4 \ln |x+5| + C \]

\[ \ln |1-y| = 2 \ln |x+5| + C \]

\[ 1-y = C(x+5)^{-4} \]

\[ y = -C(x+5)^{-4} + 1 = -\frac{C}{(x+5)^4} + 1 \]

Then $y(1) = 2$, so

\[ 2 = -C(1+5)^{-4} + 1 \]

\[ -1296 = -\frac{C}{16} \Rightarrow C = 20736 \]

\[ y = \left( \frac{1296}{(x+5)^4} + 1 \right) \]
10. A glass window at the end of a pool is perpendicular to the ground and has the shape of a trapezoid with the parallel sides parallel to the ground. The trapezoid is 6 feet high, 36 feet long on the bottom and 18 feet long on the top. The top edge of the glass window is 4 feet below the surface of the water. Recall that water has a density of 62.5 lbs/ft³. Find the total force exerted by the water on the end of the window. Show all your work and show enough work to justify your answer. **HINT:** A diagram is omitted, but it might be a good place to start organizing your information!

\[ F = p \int_0^6 \left( A_{\text{cross}} \right) (\text{position}) \]
\[ = p \int_0^6 (36-3y)(10-y) dy \]
\[ = p \int_0^6 (360-76y+3y^2) dy \]
\[ = (62.5)(1008) \]
\[ = 63,000 \text{ ft-lbs} \]

\[ A_{\text{cross}} = (2x+18) dy \]

\[ \frac{6-y}{18} = \frac{x}{4} \]
\[ x = \frac{54 - 9y}{4} \]

\[ x = 9 - \frac{3}{4}y \]

\[ 2x + 18 = 36 - 3y \]