Complex Analysis Prelim Written Exam

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Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

Warning: This is a write up of solutions, and as such there might be errors (possibly intentional... I mean they are definitely intentional), so if you want to use this material, you should check the work. If you do happen to find any (intentional) mistakes please let me know. I will likely fix them at a later date, and I can give you credit for the fix if you would like.

[1] Describe all the values of $(-1)^i$, where $i = \sqrt{-1}$.

Solution: We begin by representing $-1$ as $e^{i(\pi+2k\pi)}$ (for all $n \in \mathbb{Z}$) using Euler’s identity. Therefore

$$(-1)^i = e^{\pi+2k\pi}$$

for all $n \in \mathbb{Z}$.

[2] Write three terms of the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ in the annulus $1 < |z| < 2$.

Solution: We will begin by using partial fraction decomposition to rewrite the $f(z)$ as

$$f(z) = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-2}$$

Solving for the coefficients above, we have

$$f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$$f(z) = \frac{1}{2z} - \frac{(1/z)}{1-1/z} + \frac{1}{2(z-2)}$$
Next we have the following:

\[ f(z) = \frac{1}{2z} - \frac{1}{z - 1} + \frac{1}{2(z - 2)} \]

\[ = \frac{1}{2z} - \frac{(1/z)}{1 - (1/z)} - \frac{1}{4} \frac{1}{1 - (z/2)} \]

\[ = \frac{1}{2z} - (1/z) \left( 1 + (1/z) + (1/z)^2 + \cdots \right) - \frac{1}{4} \left( 1 + z/2 + z^2/4 + \cdots \right) \]

With that being said, here are three terms from the Laurent series:

\[ -\frac{1}{2z}, -\frac{1}{4}, -\frac{z}{8} \]

[3] Give a explicit conformal mapping from the half-disk \( \{ z : |z| < 1, \text{Re}(z) > 0 \} \) to the unit disk \( \{ z : |z| < 1 \} \).

Solution: We will achieve the desired result through a series of conformal maps. First, we define

\[ f(z) = i \frac{z + 1}{-z + 1} \]

to be the Cayley map that takes the unit disk to the upper half plane, and \( g(z) = -if(z) \). Next we define \( h(z) = z^2 \). Then the following conformal map achieves the desired result:

\[ t(z) = (f^{-1} \circ h \circ g)(z) \]

Where \( f^{-1}(z) = \frac{z - i}{z + i} \). This series of maps can be visualized below.
[4] Determine the radius of convergence of the power series for $\sqrt{z}$ expanded at $4 + 3i$.

**Solution:** Here we will take the $\sqrt{z}$ to be the principle square root of $z$ using

$$\sqrt{z} = e^{\frac{1}{2}(\ln |z| + i \arg(z))}.$$  

With that being said, we notice that this function will be holomorphic for $z \in \mathbb{C} \setminus \mathbb{R}^-$, where $\mathbb{R}^-$ denotes all real numbers less than or equal to zero. Since the power series for a function will converge in the largest disk where the function is holomorphic. This gives that the radius of convergence will be $|4 + 3i - 0| = 5$ for this function. Now suppose that there were another function, $g$ such that $g(z)^2 = z$. If $g$ had a larger radius of convergence than the function given above (when expanded about the point $4 - 3i$), then $g$ would have a power series expansion of zero with positive radius of convergence. This would imply that

$$(g(z))^2 = (c_0 + c_1 z + \frac{c_2}{2} z^2 + \cdots)^2 = c_0^2 + 2c_0 c_1 x + \cdots = z$$

If $c_0 = 0$, then this series would not converge at $z = 0$. Further if $c_0 = 0$, the linear term in the expression above would be 0, meaning that the series would only converge at $z = 0$. This would contradict the fact that $g$ had a power series expansion at zero with positive radius of convergence. Thus, the largest possible radius of convergence is achieved with the function given above, and

Radius of convergence $= 5$

[5] Evaluate $\int_{-\infty}^{\infty} e^{i\xi x} \frac{dx}{1 + x^2}$ for real $\xi$.

**Solution:** Notice that we can rewrite the above integral as the following limit

$$\lim_{R \to \infty} \int_{-R}^{R} e^{i\xi x} \frac{dx}{1 + x^2}$$

We will evaluate this integral differently depending on if $\xi$ is positive or negative.

**Case I:**

Assuming $\xi \geq 0$, we will let $\gamma_R$ be the curve consisting of the straight line from $-R$ to $R$ along the real line and the arc of the circle of radius $R$ in the upper half plane. We will now consider the integral

$$\int_{\gamma_R} e^{i\xi z} \frac{dz}{1 + z^2}$$

We begin by showing that integral over the circular arc, $C$, goes to zero as $R$ goes to infinity. For this, we will use the following trivial estimate

$$\left| \lim_{R \to \infty} \int_{C} e^{i\xi z} \frac{dz}{1 + z^2} \right| \leq \lim_{R \to \infty} \pi R \sup_{z \in C} \left| \frac{e^{i\xi z}}{1 + z^2} \right| \leq \lim_{R \to \infty} \pi R \left| \frac{e^{i\xi - \text{Im}(z)}}{R^2 - 1} \right|$$

This expression goes to zero since $\xi \geq 0$ gives a bound on $e^{i\xi}$. To see this, we can choose $R$ large enough such that $(R^2 - 1) > \frac{1}{2} R^2$, thus giving
\[
\lim_{R \to \infty} \int_C \frac{e^{i\xi z}}{1 + z^2} \, dz \leq \lim_{R \to \infty} \pi R \cdot \frac{2M_\xi}{R^2} = \lim_{R \to \infty} \frac{2\pi M_\xi}{R} = 0
\]

Therefore we have
\[
\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{1 + x^2} \, dx = \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{i\xi z}}{1 + z^2} \, dz
\]

This is equal to \(2\pi i\) times the sum of the residues enclosed by \(\gamma_R\). Since the denominator is factored as \((z + i)(z - i)\) we have only one enclosed singularity at \(z = i\). This corresponds to the residue:
\[
\text{Res}\left(\frac{e^{i\xi z}}{(z + i)(z - i)}, i\right) = \lim_{z \to i} \frac{e^{i\xi z}}{(z - i)} = \frac{e^{-\xi}}{2i}
\]

Thus
\[
\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{1 + x^2} \, dx = \pi e^{-\xi}
\]

**CASE II:**

Assuming \(\xi < 0\), the most of the arguments above hold for a curve \(\gamma_R^\prime\) in the lower half plane. However, the following addenda must be made
\[
\text{Res}\left(\frac{e^{i\xi z}}{(z + i)(z - i)}, -i\right) = \lim_{z \to -i} \frac{e^{i\xi z}}{(z - i)} = -\frac{e^\xi}{2i}
\]

Now since such a curve will give reverse orientation, we account for this by multiplying the residue by a negative one. This gives
\[
\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{1 + x^2} \, dx = \pi e^\xi
\]

Thus we can summarize our results in the following equation
\[
\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{1 + x^2} \, dx = \pi e^{-|\xi|}
\]

[6] Show that a holomorphic function \(f\) on \(\mathbb{C}\) satisfying \(|f(z)| \leq \sqrt{1 + |z|}\) for all \(z \in \mathbb{C}\) is constant.

*Solution:* Since every holomorphic function on \(\mathbb{C}\) is analytic on \(\mathbb{C}\), we may write
\[
f(z) = \sum_{n=0}^{\infty} c_n \cdot z^n,
\]
where \(c_n = f^{(n)}(0)/n!\) by Taylor’s theorem and the uniqueness of power series. Next notice that
\[
f^{(n)}(0) = \frac{1}{2\pi i} \oint_C \frac{f(w) \, dw}{w^{n+1}}
\]
for any closed rectifiable curve \( C \). Letting \( C_R \) be a circle of radius \( R \) centered at \( z = 0 \), we have

\[
|f^{(n)}(0)| = \left| \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{w^{n+1}} \, dw \right| \leq \frac{2\pi R}{2\pi i} \sup_{z \in C_R} \left| \frac{f(z)}{z^{n+1}} \right| \leq \frac{\sqrt{1+R}}{R^n} \tag{1}
\]

Next, notice that if \( n \geq 1 \), then the limit as \( R \) approaches infinity of the right hand side is zero. Since (1) holds for all \( R \), we have \( f^{(n)}(0) = 0 \) for all \( n \geq 1 \). This gives that \( c_n = 0 \) for all \( n \geq 1 \), hence \( f \) is a constant.

[7] Show that \( 4z^5 - z + 2 \) has all its zeros in the unit disk.

**Solution:** For this problem, we will use Rouché’s Theorem, and as such, we should begin by stating the version that will be used.

**Theorem 1.** Let \( f \) be holomorphic on an open set \( U \) containing \( D \), where \( \partial D \) is a simple closed path. Suppose that \( f \) does not vanish on \( \partial D \). If another holomorphic function \( g \) on \( U \) satisfies \( |f(z) - g(z)| < |f(z)| \), for all \( z \in \partial D \), then \( f \) and \( g \) have the same number of zeros inside of \( f \).

Notice that if we set \( f(z) = 4z^5 \), and \( g(z) = 4z^5 - z + 2 \), we see that \( f \) doesn’t vanish on the unit disk, and

\[
|f(z) - g(z)| = |z - 2| \leq 3 < 4 = |f(z)| \quad \text{(for \( z \) in the unit disk)}
\]

Applying Rouché’s theorem, we see that \( g \) has 5 zeros in the unit disk. Since \( g \) is a fifth degree polynomial in \( z \), it has exactly 5 zeros in \( \mathbb{C} \) by The Fundamental Theorem of Algebra.

[8] Show that there is a holomorphic function \( f(z) \) on a neighborhood of 0 so that \( f(z) = \frac{\sin z}{z} \), and determine the radius of convergence of the power series at 0.

**Solution:** First we begin by noting that

\[
\frac{\sin z}{z} = \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right)
\]

is the power series expansion of \( \frac{\sin z}{z} \) at \( z = 0 \). If we define

\[
g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots,
\]

we see that \( g \) is a holomorphic function on \( \mathbb{C} \) and \( g \) agrees with \( \frac{\sin z}{z} \) on \( \mathbb{C} \setminus \{0\} \). Further, \( g \) will share the same zeros as \( \sin(z) \) since \( g(1) = 1 \). Thus the zeros of \( g \) are at \( z = \pi + 2k\pi \) for \( k \in \mathbb{N} \).

Since \( g \) is holomorphic on the complex plane, we will define

\[
f(z) = \sqrt{g(z)} = e^{\frac{1}{2}(\log(g(z)) + i\text{Arg } g(z))}
\]

This will be well defined except when \( g(z) = 0 \). Thus the radius of convergence will be \( \pi \) since this will be the largest disk on which \( f(z) \) is holomorphic.
Describe all complex-valued harmonic functions on the annulus $1 < |z| < 2$ with extend continuously to the circle with $|z| = 2$ and take the value 0 on that circle.

**Solution:** If $f$ is harmonic, then $f_{xx} + f_{yy} = 0$. Converting to polar coordinates via $f(x, y) = f(r, \theta)$, we have

1. $f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0$

We then have the following boundary conditions:

2. $f(2, \theta) = 0$

3. $f(r, \theta) = f(r, \theta + 2\pi)$

Then our problem is to solve the Dirichlet problem on the annulus with boundary condition $f(2, \theta) = 0$. We will proceed to solve this by first writing the Fourier expansion of $f$ as

$$f(r, \theta) = c_n(r) e^{in\theta}$$

Applying the polar Laplacian, we have

$$c''_n(r) + \frac{1}{r} c'_n(r) - \frac{n^2}{r^2} c_n(r) e^{in\theta} = 0$$

Since $e^{in\theta}$ will not be identically zero (for any integer $n$), we have

$$c''_n(r) + \frac{1}{r} c'_n(r) = \frac{n^2}{r^2} c_n(r)$$

or

$$r^2 c''_n(r) + r c'_n(r) - n^2 c_n(r) = 0$$

This equation is a Cauchy-Euler differential equation, and we will solve it by using the substitution $r = e^s$ and an auxiliary function $G(s) = c_n(e^s) = c_n(r)$. This leads to the following

1. $G'(s) = d/dr[c_n(e^s)] = e^s c'_n(e^s)$

2. $G''(s) = d^2/dr^2[c_n(e^s)] = e^{2s} c''_n(e^s) + e^s c'_n(e^s)$

$$= r^2 c''_n(r) + r c'_n(r)$$

Therefore, we have if we combine this with equation 2) from (2), we obtain

$$G''(s) = n^2 G(s)$$

Next we consider cases $n = 0$ and $n > 0$. When $n = 0$ we get the solution

$$G(s) = k_0 + k_1 s = k_0 + k_1 \log(r)$$

[Note: $s = \log(r)$]

Now we consider when $n > 0$. In this case, we have solutions

$$G(s) = a_n e^{ns} + b_n e^{-ns} = a_n e^{n \log(r)} + b_n e^{-n \log(r)} = a_n r^n + b_n r^{-n} = c_n(r)$$

Therefore a general harmonic function on the annulus $1 < |z| < 2$ is of the form
\[ f(r, \theta) = k_0 + k_1 \log(r) + \sum_{n=-\infty, n \neq 0}^{\infty} (a_n r^n + b_n r^{-n})e^{in\theta} \]

Now we apply the boundary condition \( f(2, \theta) = 0 \) to see each \( c_n(2) = 0 \). This requires that \( a_n = b_n = 0 \) for \( n \neq 0 \) since the \( e^{in\theta} \) are linearly independent. Therefore

\[ f(r, \theta) = k_0 + k_1 \log(r) \quad \text{where} \quad k_0 = -k_1 \ln(2) \]

or

\[ f(z) = a \log |z| - a \ln(2) \]