1. (10 points) In this question only, you may use symbols for logical connectives.

   (a) (4 points) Write a truth table for the conditional statement \( p \lor (\sim q) \) using the symbols T/F for true/false, respectively.

   \[
   \begin{array}{c|c|c|c}
   \sim q & p \lor \sim q \\
   \hline
   T & F & T \\
   T & F & T \\
   F & T & T \\
   F & T & T \\
   \end{array}
   \]

   (b) (3 points) What is the logical relationship between \( p \lor (\sim q) \) and \( p \Rightarrow q \)? (Your answer should use at least one of the words – equivalent, converse, contrapositive, or negation)

   \[
   p \lor \sim q \text{ is logically equivalent to } q \Rightarrow p, \text{ as can be verified from the table above.}
   \]

   Then \( p \lor \sim q \) is the converse of \( p \Rightarrow q \).

   (c) (3 points) What conclusion can you draw using all of the following statements?

   - All puddings are nice.
   - This dish is a pudding.
   - No nice things are wholesome.

   This dish is not wholesome.
2. (15 points) Consider the following statement about real numbers $x$, $y$, and $z$:

There exists an $x$ such that for all $y > x$, there exists a $z$ for which $z^2 + z = y$.

(a) (6 points) Write the negation of the statement above, avoiding the use of symbols for logical quantifiers and connectives.

For all $x$, there exists a $y > x$ such that

there is no $z$ with $z^2 + z = y$.

for all $z > z^2 + z \neq y$.

(b) (9 points) Determine whether the original statement is true or false. Write “true” or “false”, and then justify your answer by proving the TRUE statement – that is, either the original statement or the negation that you wrote in (a).

The original statement is true.

To prove it, we must exhibit an $x$ such that $y > x$ guarantees a solution to $z^2 + z = y$.

In pictures, $z^2 + z - y$ is a parabola, with zeros provided that the discriminant $1 + 4y > 0$.

so we need $y > -\frac{1}{4}$ to ensure solutions.

Hence we pick $x = -\frac{1}{4}$. Then for all $y > x$,

a solution to $z^2 + z = y$ exists by quadratic formula in $z$. 
3. (8 points) Proof or counterexample – For all natural numbers $n$, if $n^3 + n + 1$ is even, then $n$ is even.

The statement is true, and we prove it by

**Contrapositive:** if $n$ is odd, then $n^3 + n + 1$ is odd.

Indeed, if $n$ odd, then we may write

$$n = 2k + 1$$

for some $k \in \mathbb{N}$.

Now we compute

$$n^3 + n + 1 = (2k + 1)^3 + (2k + 1) + 1$$

$$= 8k^3 + 6k^2 + 8k + 3$$

after some algebra

$$= 2 \cdot (4k^3 + 3k^2 + 4k + 1) + 1$$

$$= 2m + 1$$

where $m$ is a natural number. Call it $m$. Hence

So $n^3 + n + 1 = 2m + 1$

for some $m \in \mathbb{N}$

and hence is odd.
4. (12 points) Answer the following questions about relations.

(a) (6 points) Let \( A = \{1, 2, 3\} \), \( B = \{4, 5, 6\} \) and \( C = \{7, 8\} \). Let \( R \subseteq A \times C \) be the relation \( \{(1, 7), (2, 7), (3, 7), (3, 8)\} \) and \( S \subseteq B \times C \) be the relation \( \{(4, 7), (4, 8), (5, 8)\} \). Determine \( S^{-1} \circ R \).

\[
S^{-1} = \begin{cases} (7,4), (8,5) \\ (1,4), (2,4), (3,4), (3,5) \end{cases}
\]

So \( S^{-1} \circ R = \begin{cases} (1,4), (2,4), (3,4), (3,5) \end{cases} \)

(b) (6 points) Let \( A = \{1, 2, 3\} \) and let \( R \subseteq A \times A \) be the relation \( \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\} \). Which of the following adjectives, if any, apply to \( R \) – reflexive, symmetric, transitive? Explain.

\( R \) is reflexive since \( \begin{cases} (a, a) \in R \text{ for all } a \in A \\
(1,1), (2,2), (3,3) \in R \end{cases} \)

\( R \) is not symmetric. For example \( (1,2) \in R \) but \( (2,1) \notin R \).

\( R \) is transitive because \( \forall x, y, z \in A, (x,y), (y,z) \in R \implies (x,z) \in R \)

in our case:

\( (1,2), (2,3) \in R \) and \( (1,3) \in R \)

\( (1,1), (1,2) \in R \) and \( (1,2) \in R \)

\( (1,1), (1,3) \in R \) and \( (1,3) \in R \), etc...

and if \( (a,a) \in R \) there is nothing to check
5. (15 points) (5 of the 15 points will be for a writing score, as described on the course website.)

Proof or counterexample – Given any sets $A, B, C$,

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

We prove this by showing containment in both directions.

**Part I:** $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$

Given $x \in A \setminus (B \cap C)$, then $x \in A$, $x \notin (B \cap C)$.

Case $a)$: $x \in A$, $x \notin B$

$\Rightarrow x \in A \setminus B$

so then $x \in (A \setminus B) \cup (A \setminus C)$

since $A \setminus B \subseteq (A \setminus B) \cup (A \setminus C)$

So similarly, $x \in A \setminus C$.

Case $b)$: $x \in A$, $x \notin C$.

$\Rightarrow x \in A \setminus C$

so then $x \in (A \setminus B) \cup (A \setminus C)$

**Part II:** $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$

Given $x \in (A \setminus B) \cup (A \setminus C)$, we immediately make cases:

Either $x \in (A \setminus B)$ or $x \in (A \setminus C)$. (not an exclusive "or"

but these two cases exhaust all possibilities)

Case $a)$: if $x \in (A \setminus B)$

then $x \in A$, $x \notin B$

hence $x \in A \setminus (B \cap C)$

so $x \in A \setminus (B \cap C)$

Case $b)$ if $x \in (A \setminus C)$

then $x \in A$, $x \notin C$

so then $x \in A \setminus (B \cap C)$

and again $x \in A \setminus (B \cap C)$.

A smarter approach is to do half of the work, noting that the roles of $B, C$ are completely symmetric in the statement, so only one case "a" is needed.