Exploring consequences of definition of derivative: \( \text{Df}(a) \): linear term such that
\[
\lim_{h \to 0} \frac{f(a+h) - f(a) - \text{Df}(a)h}{|h|} = 0.
\]

Think of numerator as describing whether \( f(a) + \text{Df}(a) \cdot h \) is a good linear approximation to \( f(a+h) \).

If one-variable \( f \) with power series repn, then we'd subtract constant, linear terms, leaving quadratic and higher, so indeed limit is 0.

From that differentiable \( \Rightarrow \) continuous.

Directional derivatives

Properties of derivatives

1. Derivative of constant function = 0 matrix
2. Derivative of linear function is function (i.e. matrix of its linear transformation)
3. Derivative of \( f \) exists at \( a \)

\[
\Leftrightarrow \text{Derivative of } f^{(i)} \forall i = 1, \ldots, m \text{ exist at } a
\]

\[
\text{then } \text{Df}^n(a) \mathbf{x} = \begin{bmatrix}
\text{Df}^{(1)}(a) \mathbf{x} \\
\vdots \\
\text{Df}^{(m)}(a) \mathbf{x}
\end{bmatrix}
\]

4. Sums, product of \( \mathbb{R} \)-valued and \( \mathbb{R}^m \)-valued, dot products

5. All polynomials, rational functions w/ non-vanishing denom are differentiable
Additional remarks:

1. We obtain lots of properties of derivatives from properties of limits, e.g. sums, products, composition,
   \[ f \text{ differentiable} \iff f^{(i)} \text{ differentiable for } i=1,\ldots,m. \]

2. Can compute directional derivatives — viewed as generalization of partial derivatives, now not along coordinate axis but along a vector.

Define \( D_v f(a) = \lim_{t \to 0^+} \frac{f(a + tv) - f(a)}{t} \) (here \( t \) is a scalar) in \( \mathbb{R} \)

**Proposition:** If \( f \) is differentiable at \( a \), then

\[ D_v f(a) = Df(a) \cdot v \]

Jacobi matrix multi. vector \( v \)

If: Check for \( t \to 0^+ \), let you check \( t \to 0^- \). (just as easy)

Then

\[ \lim_{t \to 0^+} \frac{f(a + tv) - f(a) - Df(a) \cdot (tv)}{t} = 0. \]

Since \( f \) diff. at \( a \), but \( Df(a) \) is linear so

\[ Df(a)(tv) = t \cdot Df(a)(v) \]

\[ D_v f(a) = \lim_{t \to 0^+} \frac{f(a + tv) - f(a)}{t} = Df(a)v. \]
proof that derivative of linear map is itself:

cute idea: use uniqueness of $Df(a)$.

Indeed, $f$ is linear so \[ \lim_{h \to 0} \frac{f(a+h) - f(a) - f(h)}{|h|} = 0 \]
because nominator is identically 0. //

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other proofs of derivative same amount to clever manipulation of
difference quotient (e.g. add + subtract the same term to allow
regrouping + factoring)

and resemble their
one-variable counterparts.
chain rule: \[ g: U \rightarrow V \quad U \subseteq \mathbb{R}^n, \ V \subseteq \mathbb{R}^m \]
\[ f: V \rightarrow \mathbb{R}^p \] so \( f \circ g \) defined.

Suppose \( g \) diff. at \( a \), \( f \) diff. at \( g(a) \) then
\( f \circ g \) diff. at \( a \) write derivative
\[
D(f \circ g)(a) = Df(g(a)) \cdot Dg(a)
\]

pf. of chain rule: Kind of messy...

Composition of linear transformations (matrix mult.)

example: \( f: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)
\[
\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}
\]
\( g: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)
\[
\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \begin{bmatrix} u \cos v \\ u \sin v \end{bmatrix}
\]

\[
D(f \circ g) \begin{bmatrix} u \\ v \end{bmatrix} = Df(g(v)) \cdot Dg(v)
\]

Chain Rule

\[
g \begin{bmatrix} v \\ v \end{bmatrix} = \begin{bmatrix} u \cos v \\ u \sin v \end{bmatrix}
\]
and
\[
Df = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}
\]

So \( Df(g(v)) = \begin{bmatrix} 2u \cos v & -2u \sin v \\ 2u \sin v & 2u \cos v \end{bmatrix} \)

\[
Dg = \begin{bmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{bmatrix}
\]

Now do matrix mult.

I can also just write composition explicitly and take Jacobian.

\[
f \circ g \begin{bmatrix} u \\ v \end{bmatrix} = f \begin{bmatrix} u \cos v \\ u \sin v \end{bmatrix} = \begin{bmatrix} u^2 \cos 2v \\ u^2 \sin 2v \end{bmatrix}
\]

Check they are the same.