Warm up example: Row reduce the matrix $\begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 7 \end{bmatrix}$

$$\xrightarrow[\substack{\text{row 1 by } 1/2: \\ \frac{1}{2}R1 \\ R2}]{} \begin{bmatrix} 1 & 2 & 5 \\ 4 & 8 & 7 \end{bmatrix} \xrightarrow[\substack{R1 \\ R2-4R1}]{} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & -13 \end{bmatrix} \xrightarrow[\substack{R1 \\ -1/13\, R2}]{} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{R1-5R2 \\ R2}]{} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If this were augmented matrix $\left[\begin{array}{cc|c} 2 & 4 & 10 \\ 4 & 8 & 7 \end{array}\right]$

corresponding to $2x_1 + 4x_2 = 10$, $4x_1 + 8x_2 = 7$ then what are solutions? Ans: No sol'n. since echelon form gives:

$$1 \cdot x_1 + 2x_2 = 0$$
$$0 = 1 \leftarrow \text{Yikes!}$$

Clearly anytime we get $0 = 1$ is bad news...

In more complicated language, if we row reduce $[A \mid b]$ and the result $[\tilde{A} \mid \tilde{b}]$ has pivot in $\tilde{b}$ then NO SOLUTIONS. (pivot in $\tilde{b}$ need not be in bottom row, since could have rows of 0's below it: e.g. $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$)

Suppose $[\tilde{A} \mid \tilde{b}]$ in echelon form and $\tilde{b}$ has no pivot. 1's. Then are there solutions? On Monday, did two examples in class:

First Example:

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 1 & 2 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

unique sol'n $x_1 = 3$, $x_2 = -1$.

clear: if A reduces to $\tilde{A} = $ Id. matrix $:\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ then unique sol'n.

Similarly, if $\left[\tilde{A} \mid \tilde{b}\right] = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$, still unique sol'n.

sophisticated way to say this: have unique sol'n if every Column contains a pivotal 1.

(then $\underset{(m\times n)}{\tilde{A}}$ with $m>n$ looks like $\underset{(n\times n)}{\text{id. matrix}}$ and $m-n$ rows of 0's.)

Second Example: If row-reduced to the echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 5/17 \\ 0 & 0 & 1 & 3/17 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so reading off sol'ns: $x_3 = 3/17$

and $\underbrace{x_1 + 2x_2}_{} = 5/17$

linear equation – choose any real # for x_1 or x_2, solve for the other.

Let's pick x_2,

then $x_1 = 5/17 - 2x_2$ is uniquely determined.

Summary, for any $x_2 \in \mathbb{R}$, $\exists!\ x_1, x_3$ so that (x_1, x_2, x_3) solves system.

solutions are in one-one corresp. with points in $\mathbb{R}$. (geom. a line)

Imagine other echelon forms:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 5/17 \\ 0 & 0 & 0 & 1 & 3/17 \end{array}\right] \end{array} \rightsquigarrow \begin{array}{l} x_4 = 3/17 \\ x_1 + 2x_2 + 3x_3 = 5/17 \end{array}$$

Now have 2 free choices. Pick x_2, x_3. Determines x_1

or

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 4 & 5/17 \\ 0 & 0 & 0 & 1 & 5 & 3/17 \end{array}\right]$$

$x_4 + 5x_5 = 3/17$

$x_1 + 2x_2 + 3x_3 + 4x_5 = 5/17$

Freely pick variables in non-pivot columns (x_2, x_3, x_5), uniquely determines pivot vars (x_1, x_4)

More formally, if $[A \mid \underline{b}]$ reduces to $[\tilde{A} \mid \tilde{\underline{b}}]$ in echelon form,

then for each pivot variable x_j in j^{th} column,

if pivot occurs in k^{th} row of $\tilde{A}$, we have

equation: $$x_j = \tilde{b}_j - \sum_{\substack{i:\text{ non-}\\ \text{pivot columns}}} \tilde{a}_{k,i} x_i$$

So pick any values for non-pivot ~~columns~~ vars., uniquely determines pivot vars.

<u>Summary</u>: In augmented matrix, if $\tilde{b}$ has pivot - no sol'ns $(0=1)$

<u>else</u>: if all columns of $\tilde{A}$ are pivots $\rightsquigarrow$ unique sol'n.

if there are k non-pivot columns in $\tilde{A}$ $\rightsquigarrow$ sol'ns in 1-1 corresp. with $\mathbb{R}^k$.

(k-dim'l hyperplane)

Think about geometry of $\mathbb{R}^2$. Each equation $a_1 x_1 + a_2 x_2 = b_1$ is a line in $\mathbb{R}^2$. Intersection of two lines in general position (i.e. not parallel) is a point.

Last <u>item of</u> business: Show echelon form of A is unique!

<u>Plan</u>: Echelon form is result of row operations, and row operations preserve solution sets of system. So we just have to show that properties of solution set determine pivot locations, etc.

<u>Key observation</u>: If start with A, give echelon form B, then we can take first k columns of B, they will be echelon form for first k columns of A.

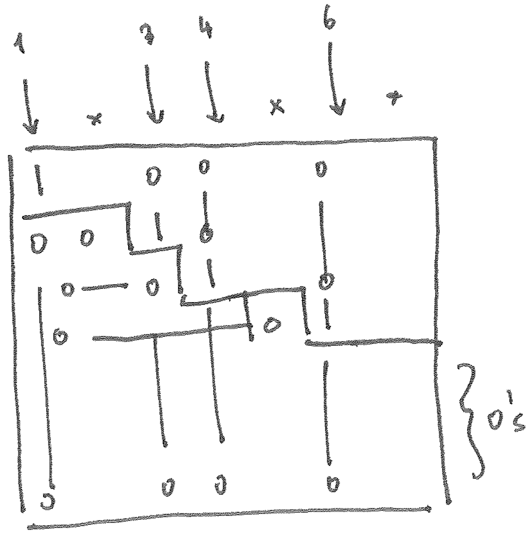
Now just make cuts at each column. Call it A_k, B_k.

think of these as mini-augmented matrices:

$$A_k = \left[\begin{array}{c|c} A_{k-1} & \begin{matrix} a_{1,k} \\ \vdots \\ a_{m,k} \end{matrix} \end{array}\right] \qquad B_k = \left[\begin{array}{c|c} B_{k-1} & \begin{matrix} b_{1,k} \\ \vdots \\ b_{m,k} \end{matrix} \end{array}\right]$$

if k is a pivot column of B, then "solution vector" has pivot in B_k, so B_k has no solutions (and A_k has no solutions)

so matrix A tells us where pivots are depending on whether mini-system A_k has solutions. This tells us alot (remember pivots are only non-zero entries in their columns in echelon form. Pivots move down to the right.)



Suppose I tell you the pivots are in columns 1, 3, 4 and 6 in matrix

To determine remaining entries, consider B_k with k non-pivotal column.

then
$$\underbrace{\left[\begin{array}{c|c} B_{k-1} & \begin{matrix} b_{1,k} \\ \vdots \\ b_{m,k} \end{matrix} \end{array}\right]}$$
has solutions. Set non-pivotal columns vars in first k columns equal to 0. then A_k has unique soln, determines $b_{1,k}, \ldots, b_{m,k}$.

these numbers characterized by solution one gets from setting non-pivot vars of B_{k-1} equal to 0.

this is subtle.
Key insight: soln to A_k with non-pivotal $x_i = 0$ is mapped to B_k with non-pivotal $x_i = 0$ by row ops.

Also prove this by induction on number of rows, knowing A, B are related by series of elementary row ops.