On Wednesday, learned that solutions to $Ax = b$ are determined by pivot columns in $[\tilde{A} \mid \tilde{b}]$, echelon form of $[A \mid b]$.  

- if $\tilde{b}$ has a pivot, no solns  
- otherwise, solns are in one-one correspondence with points in $\mathbb{R}^k$, $k =$ # of non-pivot columns.

Corollary: A reduces to id. in echelon form if and only if, for every $b$, $Ax = b$ has unique soln.

If $[A \mid b] \xrightarrow{\text{row reduce}} [I \mid \tilde{b}]$ and unique soln is $x = \tilde{b}$. (solns preserved by row reduction)

If $A$ doesn’t reduce to identity, H.H say that if $\tilde{A} \neq I_d$ then either many solutions or no solutions. This is false, as stated, for a particular choice of $b$.

One more case to rule out:

So just have to show that if $\tilde{A} = \begin{bmatrix} 1 & \cdots & \tilde{b}_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$ then for arbitrary $b$ st. $[\tilde{A} \mid b]$ has no solns.

Row reduce: look at $b_{n+1} =$ linear comb. of $b$'s. Pick $b_i$'s for arbitrary $b$.

This is desired $b$ with no solns. \( \checkmark \) equation is not 0.
We're used to thinking about matrices acting on left of vectors:

\[
A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 A(e_1) + \cdots + x_n A(e_n)
\]

columns of A

But if we multiply by row vector on left:

\[
[\begin{bmatrix} x_1 \cdots x_n \end{bmatrix}] \cdot A = x_1 \cdot (\text{row 1 of } A) + \cdots + x_n \cdot (\text{row } n \text{ of } A)
\]

For example:

\[
[\begin{bmatrix} 5 & 1 & 0 & \cdots & 0 \end{bmatrix}] \cdot A = 5 \cdot (\text{row 1 of } A) + (\text{row 2 of } A)
\]

So,

\[
[\begin{bmatrix} 1 \\ 5 \\ 1 \\ \vdots \\ 1 \end{bmatrix}] \cdot A \text{ executes the row operation replacing row 2 of } A \text{ by } 5 \cdot R_1 + R_2.
\]

Of course scalar multiplication is easy to achieve by matrix multiplication. Just use diagonal matrix:

\[
\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}
\]

to multiply row \(j\) by \(c\).

To swap rows, move 1's: e.g.,

\[
\begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}
\]

swaps first two rows.

Call these three types "elementary matrices." Compose them together to get a matrix that, when we multiply on the left by it, E

then \(E \cdot A\) is in echelon form.
Example from Wednesday: Matrix $A = \begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 7 \end{bmatrix}$

row reduced to $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

by: $\xrightarrow{\frac{1}{2} R1} \frac{1}{2} R1 \quad \xrightarrow{R2 - 4R1} \quad \frac{1}{3} R2 \quad \xrightarrow{-R1 - 5R2} R1 - 5R2$ $\quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{13} \end{bmatrix} \quad \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

Easy mnemonic for elementary matrices: Do the described op. to identity matrix.

$$E_4 E_3 E_2 E_1 \quad A = \begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 7 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{13} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{5}{13}} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{13} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-\frac{7}{26} R_1 + \frac{5}{13} R_2} \begin{bmatrix} 1 & 0 \\ -\frac{7}{26} & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 10 \\ 4 & 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

More of a useful proof tool than a method...