On Friday, we claimed the following theorem:

A is invertible (i.e. there exists C with $AC = CA = I_d$.) call C by $A^{-1}$

If and only if, for every $b \in \mathbb{R}^m$, $A x = b$ has a unique soln. (Already know this latter statement true iff $\tilde{A} = \text{In}$ identity matrix)

($\Rightarrow$) Suppose A invertible.

\[ \text{If } A x = b, \text{ then } A^{-1} A x = A^{-1} b \]

\[ \Rightarrow \quad \text{In} \]

So $x = A^{-1} b$. So showed that if $A x = b$ has solns, then unique soln is $x = A^{-1} b$.

Still have to prove solutions exist. (Check that $A^{-1} b$ is always a solution)

In our case,

\[ A (A^{-1} b) = (A A^{-1}) b = \text{In} b = b. \quad \text{(right inverse property)} \]

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Next show $(\Leftarrow)$.
What about converse: If $A\mathbf{x} = b$ has unique soln for every $b$
(i.e. $A$ reduces to $\tilde{A} = \bar{I}_n$) is it true that $A$ is invertible?

**Yes!** Pick $\tilde{b} = \tilde{e}_i$. Then $\exists \tilde{c}_i$ with $A \cdot \tilde{c}_i = \tilde{e}_i$

Make matrix $C = \begin{bmatrix} \tilde{c}_1 & \cdots & \tilde{c}_n \end{bmatrix}$ then $A \cdot C = I_n$.

$\tilde{c}_i$'s as column vectors, so $A$ has a right inverse.

**Does $A$ have $C$ as a left inverse?**

$$\begin{bmatrix} A & I_n \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} I_n & \begin{bmatrix} \tilde{e}_1 & \cdots & \tilde{e}_n \end{bmatrix} \end{bmatrix}$$

augmented matrix with $n$ vectors $\tilde{e}_i$
simultaneously

since each $A \tilde{e}_i = \tilde{e}_i$ has unique soln $\tilde{c}_i$.

With elementary matrix product $E$ such that

$$E \cdot A = I_n, \quad E \cdot I_n = C,$$ so $E = C$

and thus $C \cdot A = I_n$, a left inverse.

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We can even use this to calculate inverses...

Do simultaneously: row reduction on $A$,

keeping track of its effect

on $I_n$.

Do example in $2 \times 2$ case
Given $\tilde{v}_1, \ldots, \tilde{v}_k \in \mathbb{R}^n$, ask whether $b \in \text{Span}\{\tilde{v}_1, \ldots, \tilde{v}_k\} = \{\sum_{i=1}^k c_i \tilde{v}_i \mid c_i \in \mathbb{R}\}$.

Linear equations: Does there exist column vector $c = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$ such that $Ac = b$, where $A$ is matrix with columns $\tilde{v}_i$.

(since $Ac = c_1 \tilde{v}_1 + \cdots + c_k \tilde{v}_k$)

Linear transformations: $A$: $n \times k$ matrix $\mapsto$ linear transformation $\mathbb{R}^k \to \mathbb{R}^n$ asking whether $b$ is in the image of $T$.

To answer this question, use row reduction, solve system $Ac = b$.

E.g. $\tilde{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\tilde{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\tilde{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, is $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in $\text{Span}\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$?

Ans: Examine augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

What went wrong? Seems like 3 vectors in $\mathbb{R}^3$ should have $\text{Span}\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} = \mathbb{R}^3$. Here check $\tilde{v}_1 = -\tilde{v}_2 + 2\tilde{v}_3$.

No! System has no solution since has pivot in $\tilde{b}$.
So anything in \( \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) is actually in \( \text{Span} \{ \mathbf{v}_2, \mathbf{v}_3 \} \):

Indeed given \( c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0} \)

use relation

on \( \mathbf{v}_1 \)

\[ c_1 = c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \]

\[ c_1 = (c_2 - c_1) \mathbf{v}_2 + (2c_1 + c_3) \mathbf{v}_3 \]

\[ c_2 \]

\[ c_3 \]

\[ \mathbf{0} \in \text{Span} \{ \mathbf{v}_2, \mathbf{v}_3 \} \].

Further question: Is it unique?

(if a linear combination exists)

Example:

\[ \mathbf{d} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \]

Then \( \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{d} \) so in \( \text{Span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \)

But looking at \( \mathbf{A} \), then must be

infinitely many solutions.

Another is \( 2 \mathbf{v}_3 \). So \( \mathbf{v}_1 + \mathbf{v}_2 = 2 \mathbf{v}_3 \), i.e. \( \mathbf{v}_1 = 2 \mathbf{v}_3 - \mathbf{v}_2 \).

Plan: Remove \( \mathbf{v}_3 \). Now ask whether element in \( \text{Span} \{ \mathbf{v}_2, \mathbf{v}_3 \} \) is

unique.

\[ \mathbf{A}' = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \tilde{\mathbf{A}}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

so if a solution exists it is unique.

In fact, this shows in general that if \( \mathbf{A} \) is matrix with columns \( \mathbf{v}_1, \ldots, \mathbf{v}_k \)

then \( \mathbf{v} \in \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_k \} \) has unique linear comb.

\[ c_1 \mathbf{v}_1 + \ldots + c_k \mathbf{v}_k = \mathbf{v} \]

if and only if \( \tilde{\mathbf{A}} \) has pivots in all columns.