Return to linear transformations $T: \mathbb{R}^n \to \mathbb{R}^m \longleftrightarrow A$: $m \times n$ matrix

and discuss important subspaces associated to T.

Image (T) (or $\mathrm{Im}(T)$ for short.) $= \{ \underline{b} \in \mathbb{R}^m \mid T(\underline{x}) = \underline{b}$ for some $\underline{x} \in \mathbb{R}^n \}$

$= \{ \underline{b} \in \mathbb{R}^m \mid A \cdot \underline{x} = \underline{b} \}$

$= \{ \underline{b} \in \mathbb{R}^m \mid \underline{b} \in \mathrm{Span}\{ \text{columns of } A \} \}$

makes clear that $\mathrm{Im}(T)$ is a subspace.

Also check this directly.

Note: Subspace of $\underline{\underline{\mathbb{R}^m}}$

range space

kernel (T) (or $\mathrm{Ker}(T)$ for short)

$\|$

$\{ \underline{x} \in \mathbb{R}^n \mid T(\underline{x}) = \underline{0} \}$

$\|$

$\{ \underline{x} \in \mathbb{R}^n \mid A \cdot \underline{x} = \underline{0} \}$

$\|$

$\{ \underline{x} \in \mathbb{R}^n \mid \underline{x}$ mutually perp. to columns of $A \}$

already shown this is subspace.

So $\mathrm{Ker}(T)$ is subspace of $\mathbb{R}^n$.

Proposition: $T(\underline{x}) = \underline{b}$ (i.e. $A \cdot \underline{x} = \underline{b}$) has ~~[illegible]~~ at most one sol'n for every $\underline{b}$

if and only if $\mathrm{Ker}(T) = \underline{0}$, the trivial subspace of $\mathbb{R}^n$.

pf: We could use that $\mathrm{Ker}(T)$ determined by pivot columns of $\tilde{A}$

(whether there is 1 in each column)

But we can give simpler proof applicable more generally ...

($\Rightarrow$) By contrapositive. If $Ker(T) \neq \{\underline{0}\}$, then $T(\underline{x}) = \underline{0}$ has multiple solutions.

($\Leftarrow$) By contrapositive. If $T(\underline{x}) = \underline{b}$ has multiple solutions $\underline{x}_1, \underline{x}_2, \ldots$ $\underline{x}_1 \neq \underline{x}_2$

then $\underline{x}_1 - \underline{x}_2 \in Ker(T)$.

Find bases of $Im(T)$, $Ker(T)$. (Remember, bases not unique. Just want an algorithm that gives a basis.)

$\underline{Im(T)}$: long way: Solve $[A \mid b]$ with symbolic b's

~~if~~ if $\tilde{A}$ has rows of 0's, ensure that corresponding entry in $\tilde{b} = 0$.

Gives equations satisfied by b_i's, then find bases of these.

easier: find pivots in $\tilde{A}$, take those columns of A as basis! ($Im(T) = Span\{\text{columns of } A\}$, just have to figure out which ones to keep...)

Example: Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \rightsquigarrow \tilde{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

so ~~span~~ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis.

(example from Wed. showing ~~the~~ column vectors were linearly ~~in~~dep.)

Why does it work? Column vectors are in $Im(T)$,

since $A \cdot \vec{e}_i = i^{th}$ column of A.

To show pivot columns of A are linearly independent, solutions to $A\underline{x} = \underline{0}$ are given by (uniquely) choice of non-pivot vars. in $\tilde{A}$.

Suppose we set them all equal to 0.

E.g. if x_1, x_3, x_5 are pivot vars, x_2, x_4 non-pivots.

Set $x_2, x_4 = 0$ consider resulting equation:

$$A \cdot \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ x_5 \end{bmatrix} = \underline{0}.$$

linear combinations of pivot columns of A

Only sol'n is $x_1 = x_3 = x_5 = 0$ since, having specified non-pivot vars, sol'n is unique.

So pivot column vectors of A are linearly independent.

Why do they span $\mathrm{Im}(T)$? $\mathrm{Im}(T) = \mathrm{Span}\left\{ \begin{matrix}\text{columns} \\ \text{of } A\end{matrix} \right\}$, so

need to show non-pivot columns are linear comb. of pivot columns (hence redundant)

clear from echelon form (in fact, only need columns to left of non-pivot column: $\begin{matrix} 1 & * & * & * & * & * \\ & & 1 & * & * & * \\ & & & & 1 & * \\ & & & & & 1 \end{matrix}$)

For $\ker(T)$, reduce A to $\tilde{A}$, write sol'ns in terms of non-pivot vars. This gives basis of size equal to # of non-pivot vars.

Back to our example with $\tilde{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$: then

$$x_1 - x_3 = 0$$
$$x_2 + x_3 = 0$$

so

$$x_1 = +x_3$$
$$x_2 = -x_3$$

General sol'n: $x_3 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ so take $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ as basis.

(No magic here. Just usual approach...)

notice then, $\dim(\mathrm{Im}(T)) = \#$ of pivot columns

$\dim(\mathrm{Ker}(T)) = \#$ of non-pivot columns

so if $T: \mathbb{R}^n \to \mathbb{R}^m$ then A : $m \times n$ matrix and

$$\underbrace{\dim(\mathrm{Im}(T))}_{\text{"rank"}} + \underbrace{\dim(\mathrm{Ker}(T))}_{\text{"nullity"}} = n.$$

"Dimension Formula"

or

"Rank-Nullity Thm"

How I picture this?

Solutions of $A\underline{x} = \underline{b}$ are given by one particular solution $\underline{x_0}$ plus all elements of $\mathrm{Ker}(T)$:

If $\underline{y} \in \mathrm{Ker}(T)$ then $A(\underline{x_0} + \underline{y}) = A(\underline{x_0}) = \underline{b}$

If $\underline{x_0}, \underline{x_1}$ are two solutions,

$$A(\underline{x_0} - \underline{x_1}) = A(\underline{x_0}) - A(\underline{x_1}) = \underline{b} - \underline{b} = \underline{0}$$

So $\underline{x_0} - \underline{x_1} \in \mathrm{Ker}(T)$.

i.e. $\underline{x_1} = \underline{x_0} + \underline{y}$ for some $\underline{y} \in \mathrm{Ker}(T)$.

Corollary: $T: \mathbb{R}^n \to \mathbb{R}^n$ then

~~whe~~ $\mathrm{Im}(T) = \mathbb{R}^n$ if and only if $\mathrm{Ker}(T) = 0$.

Final result: $\mathrm{rank}(A) = \mathrm{rank}(A^T)$ T: transpose.

so span of columns has same dimension as span of rows.

pf: ~~can~~ show $\mathrm{rank}(A^T)$ is preserved under row operations.

then know $\mathrm{rank}(A^T)$ given by rows with pivots $= \#$ columns with pivots