Short Answer: Most fundamental subsets: lines, planes, etc. passing through origin.

Do examples (subspace generated by vector, $\mathbb{R}^n$ itself, solns to eqn.), non-examples (e.g. unit circle in $\mathbb{R}^2$)

LECTURE 1 STOPPED HERE!

Question 2: Our ~~[illegible]~~ geometric intuition for $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ presumes choice of mutually perpendicular vectors

call them $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, ... $\vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$. Any vector in $\mathbb{R}^n$ is a linear combination of $\vec{e}_1, \ldots, \vec{e}_n$ and no smaller subset of $\vec{e}_i$ have lin. combs. that give all vectors in $\mathbb{R}^n$

Call this a "basis" of space $\mathbb{R}^n$

Talk much more about this later,

but for now I'll use this terminology for $\{\vec{e}_1, \ldots, \vec{e}_n\}$. "standard basis"

At last, very interesting question: How to find bases for a given subset of $\mathbb{R}^n$? Even more subtle: How to find good bases for one's purposes? ("good" depends on context.)

Central idea of differential calculus:

approximate (smooth) functions locally by lines.

Generalize this to $\mathbb{R}^n$. First think about how we defined lines in one-variable calculus.

What are the natural class of maps between $\mathbb{R}^n \to \mathbb{R}^m$?

Not just sets. Have structure (can add vectors, multiply by scalars)
"vector space"

So maps should respect this structure. (VERY COMMON THEME IN MATHEMATICS!)

More precisely, maps $l : \mathbb{R}^n \to \mathbb{R}^m$ should satisfy

(a) $l(\vec{u}+\vec{v}) = l(\vec{u}) + l(\vec{v})$ any $\vec{u}, \vec{v} \in \mathbb{R}^n$

(b) $l(c \cdot \vec{u}) = c \cdot l(\vec{u})$ $c \in \mathbb{R}$, $u \in \mathbb{R}^n$

call these maps "linear transformations". If we set $\vec{u} = \vec{v} = \vec{0}$ in (a)
get $l(\vec{0}) = 2\,l(\vec{0})$
so $l(\vec{0}) = \vec{0}$.

<u>Plan</u>: use linear transformations to ~~express~~ express derivatives of functions from $\mathbb{R}^n \to \mathbb{R}^m$.

one consequence: takes subspaces to subspaces.

~~Why is this the right set of criteria?~~

We show that any linear transformation is encoded by matrix:

Any vector $\vec{v} \in \mathbb{R}^n$ can be written as $c_1 \vec{e}_1 + \cdots + c_n \vec{e}_n$ some $c_i \in \mathbb{R}$. $\vec{e}_i$: std. basis vector $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ 1 in the ith position

then using properties of linearity ⓐ and ⓑ,

$$\ell(\vec{v}) = \ell(c_1 \vec{e}_1 + \cdots + c_n \vec{e}_n) = c_1 \ell(\vec{e}_1) + \cdots + c_n \ell(\vec{e}_n)$$

so determined by $\ell(\vec{e}_1), \ldots, \ell(\vec{e}_n) \in \mathbb{R}^m$

Shorthand: make matrix A whose colums are $\ell(\vec{e}_i) := \begin{bmatrix} a_{1,i} \\ \vdots \\ a_{m,i} \end{bmatrix} \in \mathbb{R}^m$

$$A = \begin{bmatrix} | & & | \\ \ell(\vec{e}_1) & \cdots & \ell(\vec{e}_n) \\ | & & | \end{bmatrix}$$ an $m \times n$ array of real numbers

Define $A \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \overset{\text{def}}{=} \ell(\vec{e}_1) \cdot c_1 + \cdots + \ell(\vec{e}_n) \cdot c_n$ so that

$\ell(\vec{v})$ with $\vec{v} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is given by $A \cdot \vec{v}$.

Can expand right-hand side in definition; by doing the component-wise addition:

$$\ell(\vec{e}_1) \cdot c_1 + \cdots + \ell(\vec{e}_n) c_n = \begin{bmatrix} a_{1,1} \cdot c_1 + \cdots + a_{1,n} c_n \\ \vdots \\ a_{m,1} c_1 + \cdots + a_{m,n} c_n \end{bmatrix}$$

and this is more familiar rule for multiplying matrix and a vector.

Given matrix, similarly use its column vectors to define linear transformation,

so upshot: linear transforms: $\mathbb{R}^n \to \mathbb{R}^m$ $\underset{\text{1-1 correspondence}}{\longleftrightarrow}$ matrices of size m rows, n columns.

For example, if $l : \mathbb{R}^2 \to \mathbb{R}^2$ such that $l(\vec{e}_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $l(\vec{e}_2) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

then $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $l\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

This gives us idea for how to determine matrix of important geometric linear transformations. e.g. rigid motions of plane – rotations, reflections

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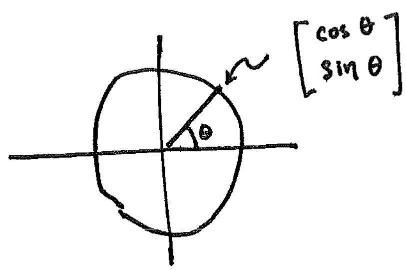
e.g. If want to rotate by θ in $\mathbb{R}^2$, then $\vec{e}_1 \longrightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

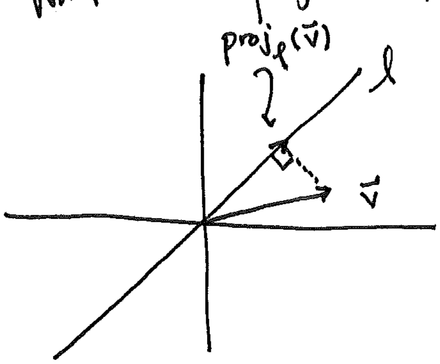
using familiar coordinates on unit circle:



$$\vec{e}_2 \longrightarrow \begin{bmatrix} \cos(\theta + \pi/2) \\ \sin(\theta + \pi/2) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

WARNING: only allowed to do this if we know in advance that the transformation is linear. Book argues this geometrically, since we're starting with geometric transformation. E.g. rotation/reflection preserve parallelogram with vertices $\vec{u}, \vec{v}, \vec{0}, \vec{u}+\vec{v}$

What about projection to a line in $\mathbb{R}^2$? Is it linear?



Simple example: projection to x-axis

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longmapsto \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$$

now compose (!) with rotation.

This is linear. Matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$)

(check directly or note