Last week, we defined dimension of subspace — the number of elements in any basis for the subspace. ( If \( V = \mathbb{F}^3 \), then \( \dim(V) = 0 \) )

This identifies \( V = \text{span} \{ v_1, \ldots, v_k \} \), a basis with \( \mathbb{R}^k \). Map: \( V \rightarrow \mathbb{R}^k \)

\[
v = \sum c_i v_i \rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}
\]

**Corollary:** If \( V \subseteq W \) and \( \dim(V) = \dim(W) \), then \( V = W \).

Used it to explore \( \text{Im}(T), \text{Ker}(T) \) for linear transformation \( T \).

Proved if \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \), linear, then

\[
\dim(\text{Im}(T)) + \dim(\text{Ker}(T)) = n
\]

**Picture in \( \mathbb{R}^2 \):** Suppose nullity = 1.

If \( T(x) = b \), then find all \( y \in \mathbb{R}^2 \) s.t. \( T(y) = b \).

- \( \text{Im}(T) \) identified with this line
- \( \text{Ker}(T) \)
- \( x + \text{Ker}(T) \)
- These lines "fill up" \( \mathbb{R}^2 \)

**Corollary:** \( T: \mathbb{R}^n \rightarrow \mathbb{R}^n \) then

\( T \) is onto \( (\text{Im}(T) = \mathbb{R}^n) \)

if and only if \( T \) is one-one \( (\text{Ker}(T) = 0) \)
Goal: polynomial interpolation of functions. Given \( f(x) \)

Say

\[
\begin{array}{c}
\text{f(x)} \\
\hline
x_0 & x_1 & x_2 \\
\end{array}
\]

Pick \( n+1 \) points, find polynomial of degree \( n \) through

\[
(x_0, f(x_0)), \ldots, (x_n, f(x_n))
\]

Here pick 3 points, find quadratic polynomial traveling through them.

Show how to do this using linear algebra. Let \( P_n \): polynomials of degree \( \leq n \) with real coeffs.

This \( P_n \) is a vector space (haven't defined this carefully axiomatically - next section - but can add, scalar multiply polynomials of degree \( n \) and remain in the set.)

Easier:

\( p \in P_n \) has form

\[
p(x) = a_0 + a_1 x + \ldots + a_n x^n.
\]

So \( P_n \) identified with \( \mathbb{R}^{n+1} = \left\{ \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \mid a_i \in \mathbb{R} \right\} \)

Consider map \( T: P_n \to \mathbb{R}^{n+1} \)

\[
p \mapsto \begin{bmatrix} p(x_0) \\ \vdots \\ p(x_n) \end{bmatrix}
\]

1. This map is linear!
2. \( \text{Ker}(T) = \{ \text{polynomials that vanish at } x_0, \ldots, x_n \} \)
3. By rank-nullity thm,

\[
\dim(\text{Im}(T)) + \dim(\text{Ker}(T)) = n+1. \quad T(p) = 0. \quad (\text{polys of } \text{deg } \leq n \text{ have at most } n \text{ roots})
\]

so given any \( \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^{n+1} \), find \( p \) with

\[
\begin{bmatrix} p(x_0) \\ \vdots \\ p(x_n) \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}
\]
To find \( p^* \) for any given \( \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \), invert \( T \). (\( T \) is invertible since it is one-one, onto.) In fact, this shows such a \( p^* \) is unique!

Example: \( x_0 = -1, \ x_1 = 0, \ x_2 = 1 \).

\[ T : \mathbb{P}_2 \rightarrow \mathbb{R}^3 \quad \text{What is its matrix?} \]

\[
p \mapsto \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} \quad + \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} \]

when \( p(x) = a_0 + a_1 x + a_2 x^2 \).

To invert \( T \), write

\[
\begin{bmatrix} T & | & 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

and keep track of row operations.

Get \( T^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \quad \text{so} \quad T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.

So given \( \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^3 \), we find polynomial through them by \( \hat{p} = T^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \) through the points \( -1, 0, 1 \).

\[
T^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ \frac{1}{2} (y_2-y_0) \\ \frac{1}{2} (y_0-\frac{3}{2}y_1+y_2) \end{bmatrix} \quad \text{this is polynomial} \quad \hat{p}(x) = y_1 + \frac{1}{2} (y_2-y_0) x + \frac{1}{2} (y_0-2y_1+y_2) x^2.
\]

This polynomial is basis of Simpson's rule, used to approximate integrals (draw picture).

To find area under quadratic, compute \( \int_{-1}^{1} \hat{p}(x) \, dx \) in general, \( \frac{1}{3} \begin{bmatrix} y_0 + 4y_1 + y_2 \end{bmatrix} \) replaced by \( \frac{b-a}{6} \) familiar from Simpson's rule.
Now you know how to generalize Simpson's rule to arbitrary degree approximations using linear algebra to find polynomial interpolations.

Trickier analysis: What is the error from this estimate?

In Simpson's rule, have error estimates in terms of 4th derivative of $f$. Higher degree approximations are sometimes called Newton-Cotes formulas.

What about error in actual interpolation? Is it true that if we take $n+1$ samples, that $p_n(x)$ converges (in some sense—e.g., pointwise) to original function $f(x)$ on closed interval $[a,b]$?

In our example, choose $x_0, \ldots, x_n$ evenly spaced.

This fails in general to converge as $n \to \infty$.

(picture in the book for classic example $\frac{1}{1+x^2}$; Runge's phenomenon, where we try evenly spaced points on $[-1,1]$ and stick bad things happen near boundary. In fact $\lim_{n \to \infty} \max_{x \in [-1,1]} \left| \frac{1}{1+x^2} - p_n(x) \right| = 0$.)

Weierstrass THEN: Continuous functions on $[a,b]$ can be uniformly approximated by polynomials, i.e., analog of above limit is 0.

$p_n(x)$ of degree $\leq n$ as $n \to \infty$.

Doesn't tell us how to pick the polynomials!

But there's a constructive proof. Don't use polynomials. Replace role of monomials $x^k$ by $x^{n-k} (1-x)^k, k = 0, \ldots, n$. in $p_n$