Last week, we defined dimension of subspace —

the number of elements in any basis for the subspace. (If $V = \{0\}$, then $\dim(V) = 0$)

this identifies $V = \mathrm{span}\{v_1, \dots, v_k\}$ $\{v_i\}$, a basis

with $\mathbb{R}^k$. Map: $V \longrightarrow \mathbb{R}^k$

$$v = \sum c_i v_i \longmapsto \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

linear ind: one-one.

span: defined on all of V

onto is by def'n of map.

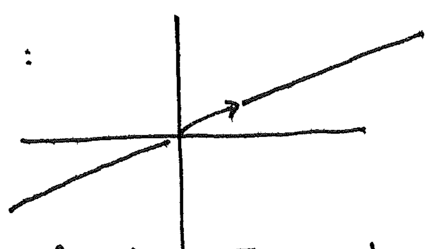
Corollary: If $V \subseteq W$ subspaces and $\dim(V) = \dim(W)$, then $V = W$.

Used it to explore $\mathrm{Im}(T)$, $\mathrm{Ker}(T)$ for linear transformation T.

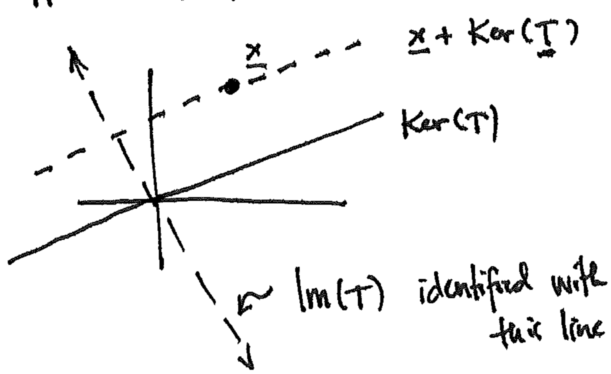
proved if $T : \mathbb{R}^n \to \mathbb{R}^m$, linear, then

$$\dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T)) = n$$

picture in $\mathbb{R}^2$: Suppose nullity $= 1$.



If $T(\underline{x}) = \underline{b}$, then find all $\underline{y} \in \mathbb{R}^2$ s.t. $T(\underline{y}) = \underline{b}$:



these lines "fill up" $\mathbb{R}^2$

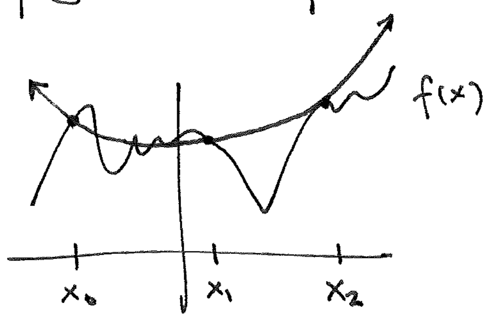
Corollary: $T : \mathbb{R}^n \to \mathbb{R}^n$ then

T is onto $(\mathrm{Im}(T) = \mathbb{R}^n)$

if and only if T is one-one

$(\mathrm{Ker}(T) = 0)$

Goal: polynomial interpolation of functions. Given $f(x)$

Say 

, sample f at some points.

Pick $n+1$ points, find polynomial of degree n through

$$(x_0, \underbrace{f(x_0)}_{y_0}), \ \dots \ , (x_n, \underbrace{f(x_n)}_{y_n})$$

Here pick 3 points, find ~~best fit~~ quadratic polynomial traveling through them.

Show how to do this using linear algebra. Let P_n: polynomials of degree $\leq n$ with real coeffs.

This P_n is a vector space (haven't defined this carefully axiomatically - next section - but can add, scalar multiply polynomials of degree n and remain in the set)

Easier:

$p \in P_n$ has form $p(x) = a_0 + a_1 x + \cdots + a_n x^n$.

So $P_n \underset{\text{identified}}{\longleftrightarrow} \mathbb{R}^{n+1} = \left\{ \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \;\middle|\; a_i \in \mathbb{R} \right\}$

Consider map $T: P_n \longrightarrow \mathbb{R}^{n+1}$

$$p \longmapsto \begin{bmatrix} p(x_0) \\ \vdots \\ p(x_n) \end{bmatrix}.$$

① This map is linear!

② $\text{Ker}(T) =$ deg $\leq n$ polynomials that vanish at $x_0, \dots, x_n$. $= 0$. (non-zero polys of deg $\leq n$ have at most n roots.)

③ By rank-nullity thm, $\text{Im}(T) = \mathbb{R}^{n+1}$ (since $\dim(\text{Im}(T)) + \dim(\text{Ker}(T)) = n+1$.)

So given any $\begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^{n+1}$, find p with $\underset{\begin{bmatrix} p(x_0) \\ \vdots \\ p(x_n) \end{bmatrix}}{T(p)} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$

To find p_* for any given $\begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$, invert T. (T is invertible since it is one-one, onto)

In fact, this shows such a p is unique!

Example: $x_0 = -1$, $x_1 = 0$, $x_2 = 1$.

$T : P_2 \longrightarrow \mathbb{R}^3$

$p \longmapsto \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

What is its matrix?

$$T \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$$

where $p(x) = a_0 + a_1 x + a_2 x^2$.

$p(-1) = a_0 - a_1 + a_2$

$p(0) = a_0$

$p(1) = a_0 + a_1 + a_2$

so $T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

To invert T,

Write $\left[\begin{array}{c|ccc} T & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array} \right]$

and keep track of row operations.

Get $T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$

So given $\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^3$, we find quadratic polynomial through them at $-1, 0, 1$ by $p = T^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$

$$T^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ \frac{1}{2}(y_2 - y_0) \\ \frac{1}{2}(y_0 - 2y_1 + y_2) \end{bmatrix}$$

this is polynomial

$$p(x) = y_1 + \frac{1}{2}(y_2 - y_0)\, x + \frac{1}{2}(y_0 - 2y_1 + y_2)\, x^2.$$

This polynomial is basis of Simpson's rule, used to approximate integrals (draw picture).

To find area under quadratic, compute $\displaystyle\int_{-1}^{1} p(x)\,dx = \frac{1}{3}\left(y_0 + 4y_1 + y_2 \right)$

familiar from Simpson's rule.

in general, $1/3$ replaced by $(b-a)/6$

Now you know how to generalize Simpson's rule to arbitrary degree approximations using linear algebra to find polynomial interpolations.

Trickier analysis: What is ~~the~~ error from this estimate?

In Simpson's rule, have error estimates in terms of 4^{th} derivative of f.

Higher degree approximations are sometimes called Newton-Cotes formulas.

What about error in actual interpolation? Is it true that if we take # of samples $\to \infty$ that $p_n(x)$ converges (in some sense - e.g. pointwise)
$n+1$

to original function $f(x)$ on closed interval $[a,b]$?

In our example, chose $x_0, \dots, x_n$ evenly spaced.

This fails in general to converge as $n \to \infty$.

(picture in the book for classic example $\frac{1}{1+x^2}$, Runge's phenomenon, where we try evenly spaced points on $[-1,1]$ and still bad things happen near boundary. In fact $\lim_{n\to\infty} \max_{x\in[-1,1]} \left| \frac{1}{1+x^2} - p_n(x) \right| = \infty$.)

Weierstrass Thm: Continuous functions on $[a,b]$ can be uniformly approximated by polynomials $p_n(x)$ of degree $\leq n$ as $n \to \infty$.

i.e. analogue of above limit is 0.

Doesn't tell us how to pick the polynomials!

But there's a constructive proof. Don't use polynomials. Replace role of monomials x^k in p_n by $x^{n-k}(1-x)^k$. $k = 0, \dots, n$.