Vector spaces can be defined abstractly –

sets of "vectors" that can be added, scalar multiplied by elt. of $\mathbb{R}$

with nice properties –

Better: Set closed under linear combinations of its elements, called "vectors"

addition is commutative, associative

scalar mult is associative, distributive.

— mention examples: $P^{(n)}$: polys of degree $\leq n$, $\mathrm{Mat}_{n\times m}$ —

If vector space is finite dimensional, then always pick basis $v_1, \dots, v_k$ (say dimension k)

and identify V with $\mathbb{R}^k$:

$$T: V \longrightarrow \mathbb{R}^k \qquad (*)$$

$$\sum_i c_i v_i \longmapsto \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = c_1\vec{e}_1 + \cdots c_k\vec{e}_k$$

Proper notion of "identify": We say V, W are "isomorphic" if

there exists a linear $\underbrace{\text{bijection}}_{\text{one-one corresp. of sets}}$ $T: V \longrightarrow W$.

Important point: Vector space has more structure than ordinary set.

So bijection should reflect additional structure – it should be linear

i.e. $T(v_1 + v_2) = T(v_1) + T(v_2)$ so takes sums in V to sums in W, respecting law of addition. etc.

check $(*)$ is isomorphism of vector spaces

$$T(v+w) = T\left(\sum_i c_i v_i + \sum_i b_i v_i\right)$$

$$= \sum c_i T(v_i) + \sum_i b_i T(v_i)$$

linearity of T

$$= \sum (c_i + b_i) T(v_i) = T(v) + T(w).$$

leave you to check scalar mult. (immed. from linearity of T)

~~[illegible]~~ Fact that (*) is bijection is clear from definition, so it is an isom. of vector spaces.

Book calls (*) the "abstract to concrete" map since it realizes abstract vector space as $\mathbb{R}^k$, with standard basis.

Note: if you don't know in advance the dimension of V, can try to establish bijection like T for some $\mathbb{R}^k$, using choice of $v_1, \dots, v_p \in V$.

Not all vector spaces are finite dimensional, can't choose basis and work with linear transformations as matrices...

E.g. $C([a,b])$: Continuous functions on interval $[a,b]$.

pf: suppose that we could find finite basis $f_1, \dots, f_n$ so that any $f \in C([a,b])$ could be expressed as

$$f(x) = \sum_{i=1}^{n} c_i f_i(x)$$

Pick $n+1$ pts x_j. We can find an f with $f(x_j) =: y_j$ for any $y_1, \dots, y_{n+1}$. (lots of freedom in $C[a,b]$)

Claim: $\exists$ choice of y_j's s.t. the linear system ($n+1$ equ's in n unknowns) is not solvable in c_k's:

$$y_j = f(x_j) = \sum_{k=1}^{n} c_k f_k(x_j) .$$

Contradiction!

just row reduce, get bottom row of 0's. this gives conditions on y_j's. Pick accordingly.

We are used to having $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ mean $2\vec{e}_1 + 1\vec{e}_2$ in $\mathbb{R}^2$.

But we could also use it for any basis $\{v_1, v_2\}$ in $\mathbb{R}^2$. new definition.

e.g. $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2v_1 + v_2$

is the point $(3, 1)$ in $\mathbb{R}^2$ $= 3e_1 + e_2$

If $w_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ another basis;

I can convert between basis by linear transformation

$$\mathbb{C}: \begin{aligned} w_1 &\mapsto w_1 \\ w_2 &\mapsto w_2 \end{aligned}$$

$w_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = v_1 + v_2$ → column 1 of T

$w_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2v_1 + v_2$ → column 2 of T

so $\mathbb{C} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.

check $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ means w_1 in $\{w_1, w_2\}$ - basis.

want $\mathbb{C}$ s.t. $\mathbb{C}(\underline{a}) = \underline{b}$

where $\sum a_i w_i = \sum b_j v_j$

expand $w_i = \sum c_j v_j$

$$\mathbb{C} \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{= w_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1 + v_2$$

Example 2: Map to standard basis.

$\mathbb{C}: w_1, w_2 \mapsto \vec{e}_1, \vec{e}_2$. $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$w_1 = 2e_1 + 0e_2$

$w_2 = 3e_1 + e_2$

$\mathbb{C} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$.

Then compose with

$D: \vec{e}_1, \vec{e}_2 \mapsto v_1, v_2$

$D = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} +1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$$D \cdot C = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

✓

Best basis for a transformation T? Eigenbasis!

Look for basis $\{v_1, v_2\}$ such that $T(v_1) = \lambda_1 v_1$, $T(v_2) = \lambda_2 v_2$ some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Then, with respect to basis $\{v_1, v_2\}$,

$$T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Very easy to understand geometrically, easy to do matrix ops with, ...

How can we find such v_1? $\lambda_1 v_1 = \lambda_1 I_2 \cdot v_1$ so

$T(v_1) = \lambda_1 v_1 \iff (T - \lambda_1 I_2) \cdot v_1 = 0$

Ask when $(T - \lambda I_2)$ has non-trivial kernel.

2×2 matrix has non-triv. kernel when bottom row is all 0's.

(i.e. $T - \lambda I_2 \neq \mathrm{Id}$. i.e. $T - \lambda I_2$ not invertible.

For 2×2 matrices, this is just when $\det(T - \lambda I_2) = 0$.

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \lambda I_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ so } T - \lambda I_2 = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$\det(T - \lambda I_2) = (a-\lambda)(d-\lambda) - bc$$

: quadratic equation in λ. Find roots to make it $= 0$.

<u>Example</u>: $T = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

$$\det(T - \lambda I_2) = (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1)$$

So kernel non-trivial when $\lambda = 5, 1$.

if $\lambda = 1$: $T - I_2 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ with $v_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in $\ker(T - I_2)$

if $\lambda = 5$: $T - 5I_2 = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$ with $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in $\ker(T - 5I_2)$