On Wednesday, discussing inverse functions.

Have $f : \mathbb{R}^n \to \mathbb{R}^n$. Seek $g$ s.t. $g(f(x)) = x$, $f(g(y)) = y$.

Example: $f : \mathbb{R} \to \mathbb{R}$ with graph:

Not invertible, since not one-one.

(invertible $\iff$ one-one and onto)

But we could ask for invertibility on subset $(a, b) \subseteq \mathbb{R}$.

Avoid $x$ maxima/minima of the function.

(places where $f'(x) = 0$, i.e. derivative is not invertible!)

If we want to know value of $g(7)$, e.g. solve for when $f(x) = 7$, i.e. $f(x) - 7 = 0$ can do this by Newton's method.

Inverse function theorem (qualitative version)

If $f : U \to \mathbb{R}^n$ continuously differentiable (i.e. first partials continuous)

If $Df(x_0)$ invertible, then $f$ is invertible, with differentiable inverse, on an open neighborhood of $f(x_0)$.

Intuition: $Df(x_0)$ invertible if det $[Df(x_0)] \neq 0$.

(from $\mathbb{R}^2 \to \mathbb{R}^2$) so if continuous, we can move nearby $x_0$ and $Df(x)$ will (say to $x$) still have non-zero det.

Want to sharpen, then prove, inverse function theorem, but first do example from the book.
Idea: Use inverse function theorem to study image of function.

If $f: \mathbb{R} \to \mathbb{R}$ (continuous), then image of $f$ is connected.

If $f: \mathbb{R}^2 \to \mathbb{R}^2$, connected sets much more interesting.

\[ F(\theta, \phi) = \frac{1}{2} \begin{pmatrix} 3 \cos \theta + \cos \phi + 10 \\ 3 \sin \theta + \sin \phi \end{pmatrix} \]

(midpoints of lines connecting pts. on two circles)

Key idea: If $F$ invertible in nbhd. of $F(\theta_0, \phi_0)$, then all pts in nbhd are in the image of $F$. Translating, if $DF(\theta_0, \phi_0)$ is invertible, then $(\theta_0, \phi_0)$ is an interior point in the image.

Thus boundary of image should be at points where $DF(\theta_0, \phi_0)$ is not invertible (i.e. det $DF(\theta_0, \phi_0) = 0$)

We compute $DF(\theta, \phi) = \begin{pmatrix} -\frac{3}{2} \sin \theta & -\frac{1}{2} \sin \phi \\ \frac{3}{2} \cos \theta & \frac{1}{2} \cos \phi \end{pmatrix}$ with det = $-\frac{3}{4} \left( \sin \theta \cos \phi - \cos \theta \sin \phi \right)$

so det = 0 when $\theta = \phi$ or $\phi + \pi$.

When $\theta = \phi$:

\[ F(\theta) = \frac{1}{2} \begin{pmatrix} 3 \cos \theta + \cos \phi + 10 \\ 3 \sin \theta + \sin \phi \end{pmatrix} = \begin{pmatrix} 2 \cos \theta + 5 \\ 2 \sin \theta \end{pmatrix} \] circle of radius 2 at (5, 0)

When $\phi = \theta - \pi$:

use that $\cos(\theta - \pi) = -\cos \theta$ \hspace{1cm} $\sin(\theta - \pi) = -\sin \theta$

\[ F\left(\begin{pmatrix} \theta \\ \theta - \pi \end{pmatrix}\right) = \begin{pmatrix} \cos \theta + 5 \\ \sin \theta \end{pmatrix} \] circle of radius 1 at (5, 0)
Looking for a connected region of \( \mathbb{R}^2 \) whose boundary is a subset of annulus only two possibilities:
- one of circles or
- the annulus.

Argue that \((5, 0)\) can't be in image, so must be annulus.

If \((5, 0)\) is midpoint of segment with point on \(C_2\) : circle of radius 1 at \((10, 0)\)
then other point is on the circle \(C_3\) of radius 1 at \((0, 0)\).

Quantitative Version of Inverse Function Theorem: \( f : U \to \mathbb{R}^n \) cont. diff

Give radius \(R\) for which \( f\) inverse function to \(f\) at \(f(x_0)\) is defined on \(B_R(f(x_0))\) : Find \(R\) s.t.

1. \[ U_0 := B_{2R} \cap \left\{ Df(x_0)^{-1} \cdot x \right\} \subseteq U \]

2. On \(U_0\), \(Df\) is Lipschitz with ratio \[ \frac{1}{2R |Df(x_0)|^2} \]

then \( \exists! \) continuously diff. \( g : B_R(f(x_0)) \to U_0 \)

\[ g(f(x)) = x_0 \text{ and } f(g(y)) = y \quad \forall y \in B_R(f(x_0)) \]

Important, since \( f \) may map points of \( U \) outside \( B_R(f(x_0)) \),

Hence not \( g(f(x)) = x \quad \forall x \in U \)
Construct inverse via Kantorovich's theorem, with initial point $x_0$ s.t. $f(x_0) - y = 0$, i.e., given $y \in \mathcal{V} \subseteq \text{range space}$, use Newton's method to find root $x$ of $f(x) - y = 0$.

Its Jacobian is just $[Df(x_0)]$ since $y$ fixed const.

So $\| u \| = \left\| - [Df(x_0)]^{-1} \cdot (\frac{f(x_0) - y}{y}) \right\|$

$\Rightarrow \| u \| \leq \left\| [Df(x_0)]^{-1} \right\| \cdot R \quad (\ast)$

where $R$: radius of ball defining $\mathcal{V}$.

We've set up $U_0$ to have radius $2$ - (right-hand side of $(\ast)$)

so $x_1 = x_0 + u$ and ball at $x_1$ with radius $\| u \|$ still contained in $U_0$.

For Kantorovich's thm to hold, need

$\| f(x_0) - y \| \cdot \| Df(x_0)^{-1} \|^2 M \leq \frac{1}{2}.$

But we chose Lipschitz ratio $M$ precisely so that this holds, noting $\| f(x_0) - y \| \leq R$.

$\exists x_0 \in U_0$ such that $f(x) - y = 0$, call the root $x = f^{-1}(y)$.

Since $f(f^{-1}(y)) = y$.

Works for all $y \in \mathcal{V}$ according to proof.

In particular, if $f^{-1}(y_0) = x_0$, still need to show resulting function $f^{-1}$ is continuously diff.
Classic warning example: It may be that $Df(x) \neq 0 \forall x$.

Then inverse function theorem says local inverse exists, but not necessarily a global inverse. Example: $\mathbb{R}^2 \to \mathbb{R}^2$ 
\[ (x, y) \mapsto \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix} = e^x \neq 0 \forall x \]

but clearly not one-one since $\cos, \sin$ are $2\pi$ periodic.

Corollary of inverse function theorem:

We can compute derivative of $f^{-1}$ using chain rule:

(now that we know it is differentiable)

\[ [Df^{-1}(y)] = [Df(f^{-1}(y))]^{-1} \quad \text{since} \quad f \circ f^{-1}(y) = y. \]

To really finish pf. of inverse function theorem, need to show

1. $f$ is injective on $U_0$, thus $f^{-1}$ unique inverse.
2. $f^{-1}$ continuous (messy set of inequalities, see Appendix A.7)
3. $f^{-1}$ differentiable
4. $f^{-1}$ has continuous partials.

One theme: change coordinates and rescale so that $f$ analyzed at $0$ not $x_0$, with $Df(0) = \text{Id}$. Makes analyzing inequalities much easier.