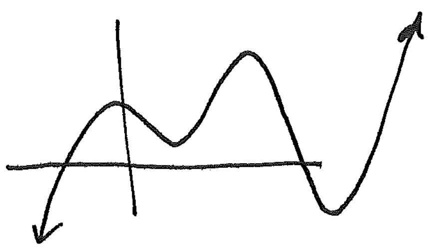
On Wednesday, discussing inverse functions.

Have $f: \mathbb{R}^n \to \mathbb{R}^n$. Seek g s.t. $g(f(\underline{x})) = \underline{x}$.
$f(g(y)) = y$

<u>Example</u>: $f: \mathbb{R} \to \mathbb{R}$ with graph:



Not invertible, since not one-one.

(invertible $\iff$ one-one and onto)

But we could ask for invertibility on subset $(a,b) \in \mathbb{R}$.

Avoid maxima/minima of the function.

(places where $f'(x) = 0$, i.e. derivative is not invertible!)

If we want to know value of $g(7)$, e.g. solve for when $f(x) = 7$
i.e $f(x) - 7 = 0$ can do this by Newton's method.

<u>Inverse Function Thm</u> (Qualitative Version)

If $f: U \to \mathbb{R}^n$ continuously differentiable (i.e. first partials continuous)

If $Df(\underline{x}_0)$ invertible, then f is invertible, with differentiable inverse,

on an open neighborhood of $f(\underline{x}_0)$.

<u>Intuition</u>: $Df(\underline{x}_0)$ invertible if $\det[Df(\underline{x}_0)] \neq 0$.
(from $\mathbb{R}^2 \to \mathbb{R}^2$)

so if continuous, we can move nearby $\underline{x}_0$ (say to $\underline{x}$) and $Df(\underline{x})$ will still have non-zero det.

Want to sharpen, then prove, inverse function thm, but first do example from the book.

<u>Idea</u>: Use inverse function theorem to study image of function.

If $f: \mathbb{R} \to \mathbb{R}$ (continuous), then image of f is connected
since
diff.
region of $\mathbb{R}$, so need to find max / min.
(i.e. search for where $f' = 0$)

If $f: \mathbb{R}^2 \to \mathbb{R}^2$, connected sets much more interesting.

e.g. $F\begin{pmatrix}\theta \\ \phi\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3\cos\theta + \cos\phi + 10 \\ 3\sin\theta + \sin\phi \end{pmatrix}$

(midpoints of lines connecting pts. on two circles)

<u>Key idea</u>: If F invertible in nbhd. of $F\begin{pmatrix}\theta_0 \\ \phi_0\end{pmatrix}$, then all pts in nbhd are in the image of F. Translating, if $DF\begin{pmatrix}\theta_0 \\ \phi_0\end{pmatrix}$ is invertible, then $\begin{pmatrix}\theta_0 \\ \phi_0\end{pmatrix}$ is an interior point in the image.

Thus boundary of image should be at points where $DF\begin{pmatrix}\theta_0 \\ \phi_0\end{pmatrix}$ is not invertible (i.e. $\det DF\begin{pmatrix}\theta_0 \\ \phi_0\end{pmatrix} = 0$)

We compute $DF\begin{pmatrix}\theta \\ \phi\end{pmatrix} = \begin{bmatrix} -\frac{3}{2}\sin\theta & -\frac{1}{2}\sin\phi \\ \frac{3}{2}\cos\theta & \frac{1}{2}\cos\phi \end{bmatrix}$ with $\det = -\frac{3}{4}(\sin\theta\cos\phi - \cos\theta\sin\phi)$

$= -\frac{3}{4}\sin(\theta - \phi)$

so $\det = 0$ when $\theta = \phi$ or $\phi + \pi$.

When $\theta = \phi$: $F\begin{pmatrix}\theta \\ \theta\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3\cos\theta + \cos\theta + 10 \\ 3\sin\theta + \sin\theta \end{pmatrix}$

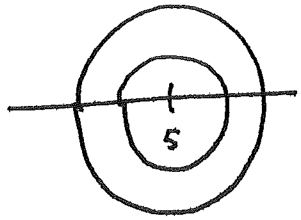
$= \begin{pmatrix} 2\cos\theta + 5 \\ 2\sin\theta \end{pmatrix} \leftarrow$ circle of radius 2 at $(5,0)$

When $\phi = \theta - \pi$:

use that $\cos(\theta - \pi) = -\cos\theta$
$\sin(\theta - \pi) = -\sin\theta$

$F\begin{pmatrix}\theta \\ \theta - \pi\end{pmatrix} = \begin{pmatrix} \cos\theta + 5 \\ \sin\theta \end{pmatrix}$ circle of radius 1 at $(5,0)$

Looking for a connected region of $\mathbb{R}^2$ whose boundary is a subset of annulus



only two possibilities: one of circles or the annulus.

Argue that $(5,0)$ can't be in image, so must be annulus.

if $(5,0)$ is midpoint of segment with point on C_2: circle of radius 1 at $(10,0)$

then other point is on the circle C_3 of radius 1 at $(0,0)$.

Quantitative Version of Inverse Function Thm: $f : \mathcal{U} \to \mathbb{R}^n$ cont. diff

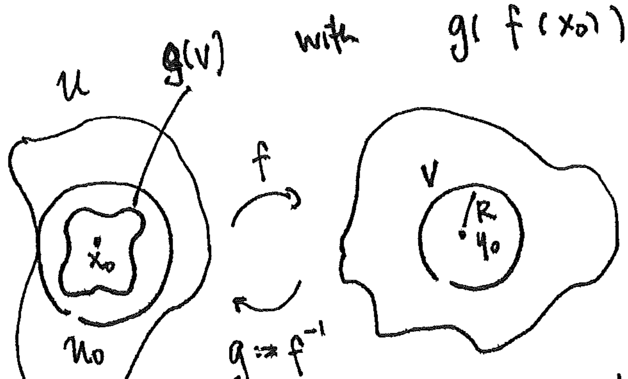
Give radius R for which inverse function to f at $f(\underline{x}_0)$ is defined on $B_R(f(\underline{x}_0))$: Find R s.t.

① $\mathcal{U}_0 := B_{2R|Df(\underline{x}_0)^{-1}|}(\underline{x}_0) \subseteq \mathcal{U}$

② On $\mathcal{U}_0$, Df is Lipschitz with ratio $\dfrac{1}{2R\,|Df(\underline{x}_0)^{-1}|^2}$

then $\exists!$ continuously diff. $g : B_R f(\underline{x}_0) \to \mathcal{U}_0$

with $g(f(x_0)) = x_0$ and $f(g(y)) = y \;\; \forall\, y \in \underbrace{B_R(f(\underline{x}_0))}_{V}$



Important, since f may map points of $\mathcal{U}$ outside $B_R(f(\underline{x}_0))$.

and even points of $\mathcal{U}_0$

Hence not $g(f(\underline{x})) = \underline{x} \;\; \forall\, \underline{x} \in \mathcal{U}$

Pf of inverse function thm:

Construct inverse via Kantorovich's theorem, with initial point x_0 s.t. $f(x_0) = y_0$

i.e. given $y \in V \subseteq$ range space $\mathbb{R}^n$, use Newton's method to find root $\underline{x}$ of $f(\underline{x}) - \underline{y} = 0$.

Its Jacobian is just $[Df(x)]$ since y fixed const.

so $\underline{r}_0 = -[Df(x_0)]^{-1} \cdot (\underbrace{f(x_0)}_{y_0} - y)$

(Maybe $\underline{r}_0(y)$ is better notation, since want to understand dependence on y)

$$\Rightarrow |\underline{r}_0| \leq \left| [Df(x_0)]^{-1} \right| \cdot R \qquad (*)$$

where R: radius of ball defining V.

We've set up U_0 to have radius $2 \cdot$ (right-hand side of $(*)$)

so $x_1 = x_0 + \underline{r}_0$ ~~[illegible]~~ and ball at x_1 with radius $\underline{r}_0$ still contained in U_0

For Kantorovich's thm to hold, need

$$|f(x_0) - y| \cdot |Df(x_0)^{-1}|^2 M \leq 1/2$$

Hence if Df Lipschitz on U_0, this suffices.

But we chose Lipschitz ratio M precisely so that this holds, noting $|f(x_0) - y| \leq R$.

$\{x_n\} \longrightarrow$ root of $f(\underline{x}) - y$, call the root $\underline{x} = f^{-1}(y)$.

since $f(f^{-1}(y)) = y$

Works for all $y \in V$ according to proof.

In particular $f^{-1}(y_0) = x_0$.

Still need to show resulting function f^{-1} is continuously diff.

Classic warning example: It may be that $Df(\underline{x}) \neq 0$ $\forall \underline{x}$, then Inverse function theorem says local inverse exists, but not necessarily a global inverse.

Example: $\mathbb{R}^2 \to \mathbb{R}^2$

$$(x,y) \mapsto \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix}$$

$\det(Df(x,y)) = e^x \neq 0$ $\forall x$

but clearly not one-one since cos, sin are 2π periodic.

Corollary of Inverse Function Thm:

We can compute derivative of f^{-1} using chain rule: (now that we know it is differentiable)

$$[Df^{-1}(y)] = [Df(f^{-1}(y))]^{-1} \quad \text{since } f \circ f^{-1}(y) = y.$$

To really finish pf. of inverse function thm, need to show

① f is injective on U_0, thus f^{-1} unique inverse.

② f^{-1} continuous (messy set of inequalities. see Appendix A.7

③ f^{-1} differentiable

④ f^{-1} has continuous partials.

one theme: change coordinates and rescale so that f analyzed at $\underline{0}$ not $\underline{x_0}$, with $Df(\underline{0}) = Id$. Makes analyzing inequalities much easier.