Friday, we discussed statements of inverse function theorem.

**Qualitative:** If \( f: U \to \mathbb{R}^n \) is contin. diff., \( x_0 \in U \) with \( Df(x_0) \)
invertible, then \( f \) is invertible with diff. inverse in a nbhd. of \( f(x_0) \)

That is, the equation \( f(x) = y \) is solvable in \( x \) for all \( y \in B_r(f(x_0)) \)

Look at 1-variable pictures: \( f: \mathbb{R}^1 \to \mathbb{R}^1 \)

![Graph of a function](image)

- Inverse function fails since \( f'(x_0) = 0 \)
- Not invertible.
- Indeed, we can't solve for \( f(x) = y \) if \( y < y_0 \).

Another point of view: \( f \) satisfying above conditions
(invertible derivative) is behaving like a linear
system, namely the system \([Df(x_0)]\), in nbhd. of \( x_0 \).

**Quantitative version:** Don't want to restate this. Just remind you
that we place extra condition that \( Df \) Lipschitz (stronger than \( Df \) continuous)
with just right Lipschitz ratio to apply Kantorovich’s Thm to

\[ f(x) - y = 0 \quad \text{for any } y \in B_r(y_0) \quad \text{with initial guess } x_0. \]

Remarks: 1. Conditions of theorem show \( x_0, x_1, \ldots \) converges to root of \( f(x) - y = 0 \) so its limit is \( f^{-1}(y) \).
1. cont. - Still need to show $f^{-1}$ differentiable with continuous partials. (prove some inequalities - heart of analysis in these arguments. See p. in appendix)

2. If $Df(x)$ invertible for all $x \in U$, still might not have global inverse.

Example: $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$(x, y) \mapsto \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix}$$

$\det(Df(x,y)) = e^{2x} \neq 0 \forall (x,y) \in \mathbb{R}^2$. But $f$ is not one-one since $\cos y, \sin y$ periodic of period $2\pi$.

3. Corollary: Now compute derivative of $f^{-1}$ using chain rule, (now that we know $f^{-1}$ defined and differentiable, under these assumptions)

$$[Df^{-1}(y)] = [Df(f^{-1}(y))]^{-1}$$

since $f \circ f^{-1}(y) = y$.

Classic example: $e^{\ln x} = x$ where we think of $\ln x$ as inverse of $e^x$.

Chain rule on left:

$$\frac{d}{dx}(e^{\ln x}) = e^{\ln x} \cdot \frac{d}{dx} \ln x$$

While $\frac{d}{dx} (x) = 1$, so $\frac{d}{dx} (\ln x) = \frac{1}{x}$.

So far, have non-linear analogue of Theorem in Section 2.2. If $A$

reduces to $\mathbf{I}$ in echelon form, then system $A\mathbf{x} = \mathbf{b}$ has unique solution, for every $\mathbf{b}$. "Locally, $f$ is behaving like linear function"
Now finish with non-linear version of general theorem on solns to

$$A\mathbf{x} = \mathbf{b} \quad \mathbf{b} \in \mathbb{R}^n, \; \tilde{A} \text{ has } n \text{ pivot columns, } m \text{ non-pivot columns}.$$ 

Then can freely choose \( m \) non-pivot variables, and these determine unique choice of \( n \) pivot variables giving a soln.

(Thm. 2.2.1, part 2b.)

Recall there’s ambiguity in these terms. Pivotal/non-pivotal variables just depend on arbitrary ordering. (we list \( x_1 \) before \( x_2 \), etc.)

But number of them is \( n - m \) each is intrinsic.

Generalization to non-linear equations is called “Implicit Function Theorem” — met this before, e.g.

unit circle: \( x^2 + y^2 = 1 \)

Rewrite as: \( x^2 + y^2 - 1 = 0 \).

View \( x \) as independent variable, write \( y = \sqrt{1 - x^2} \). Defines a function on some nbhd. of \((x_0, y_0)\) provided \((x_0, y_0) \neq (-1, 0), (1, 0)\).

Implicit function Thm.: \( f: U \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^n \) continuously diff.

s.t. \( f(x) = 0 \), and \( \text{Df}(x): \mathbb{R}^{n+m} \to \mathbb{R}^n \) is onto.

(hence write \( n \) pivot vars, \( m \) non-pivot vars)

Then \( f \) nbhd. of \( x \)

in which \( f(x) = 0 \) defines \( n \) dependent vars in terms of \( m \) independent vars, via function \( g \).
In our example $f(x, y) = x^2 + y^2 - 1 : \mathbb{R}^2 \to \mathbb{R}$

\[ Df = 1 \times 2 \text{ matrix } \begin{bmatrix} 2a, 2b \end{bmatrix} \text{ at } (a, b). = 1, \text{ on unit circle.} \]

If we want $x$ independent, $y$ dependent, $b \neq 0$.

Better: view $y$ as a function of $x$.

There is a quantitative version of the implicit function theorem with Lipschitz satisfying precise inequality for Kantrovich.

**Corollary:** By chain rule,

\[
[Dg(b)] = \left[ D_1 F(c), \ldots, D_m F(c) \right]^{-1} \left[ \begin{array}{c} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{array} \right]
\]

Remark: If we write our solution $s = (a) \in \mathbb{R}^n$ to $F(s)$ with $s = \left( \begin{array}{c} a \\ b \end{array} \right) \in \mathbb{R}^n$ and $c = (b) \in \mathbb{R}^m$ then the implicit function $g: \mathbb{R}^m \to \mathbb{R}^n$ is defined on a neighborhood of $b \in \mathbb{R}^m$ with $g(b) = a$ and $f(g(c)) = 0$.

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Example from Krantz-Parke on Implicit Function Theorem:

\[ y^5 + 16y - 32x^3 + 32x = 0. \]

Check from graph that it appears we can define \( y \) as a function of \( x \),
(but not \( x \) as a function of \( y \)).

If \( y \) is derivative in \( y \) for any \( x \) fixed \( x \) is

\[ 5y^4 + 16 > 0. \]

So monotonically increasing.

As \( y \to \pm \infty \), \( \text{LHS} \to \pm \infty \).

Apply intermediate value theorem:

\( \exists \, y \text{ s.t. } \text{LHS} = 0. \)
idea of implicit function theorem:

Suppose given $F : U \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^n$ with $F(\xi) = 0$. and $DF(\xi)$ onto. Split up domain $\mathbb{R}^{n+m}$ into $x \in \mathbb{R}^m$ (non-pivot) and $y \in \mathbb{R}^n$ (pivot).

Consider new function $f$ made from $F$: $f : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$

$$f(x, y) = \left[ \begin{array}{c} x \\ \frac{y}{w} \\ F(x, y) \end{array} \right] : \mathbb{R}^{n+m}$$

Use inverse function theorem at $(x_0, y_0) = \xi$ with $F(\xi) = 0$.

so construct inverse from nbhd. of $[x_0, 0]$.

Call it $g : \left[ \begin{array}{c} x_0 \\ 0 \end{array} \right] \to \left[ \begin{array}{c} x \\ \phi(x, 1) \end{array} \right]$ where $\phi$ is defined according to this inverse $g$.

This is desired implicit function.