Even the most basic curves, surfaces, shapes may not be everywhere in their domain expressible as functions of given collection of independent variables.

Unit circle is expressible as function of one var. on half-circle, but not over whole circle. In some cases, we must be satisfied with expressing object locally as a function of some indep. variables.

"locally" means in a neighborhood of each point. "neighborhood" for $x_0$ means (open) set containing $B_{\varepsilon}(x_0)$ for some $\varepsilon > 0$.

(By contrast "globally" refers to a property of function on entire domain. Met these terms in 1-var. calculus when discussing "local vs. global " extrema )

Basic point: When a hyper-surface looks locally like a differentiable function, all the tools of calculus apply.

Motivates our definition of "manifold" — Hubbard’s definition is different than most everyone else.

Definition: A subset of $\mathbb{R}^n$, $M$, is a smooth $k$-dimensional manifold if, for every $x \in M$,

$\exists$ nbhd $V \subset \mathbb{R}^n$ such that $V \cap M$ is the graph of a $C^1$ function $f$ of $k$ variables.
Picture: \( \mathbb{R}^2 \). \( M \): 1 dim'l manifold.

\[ M \text{ looks like function of } x \text{ on } V \cap M \]

Means we can write points in \( V \cap M \)

as \[ \begin{bmatrix} x \\ \phi(x) \end{bmatrix} \]

for some \( x \).

If this reminds you of implicit function theorem, good since that will be a major method for finding manifolds.

How to check the unit circle is 1-dim' l manifold?

(In this case, solve explicitly of any \((x_0, y_0) \text{ s.t. } x^2 + y^2 - 1 = 0\))

Example 2: Surfaces in \( \mathbb{R}^3 \): \( M \): 2-dim'l manifold.

\[ \text{z} \text{ flying carpet bending back on itself.} \]

Write points in \( V \cap M \)

as \[ \begin{bmatrix} x \\ y \\ \phi(x, y) \end{bmatrix} \]

a \( C^1 \) function.

It we chose point on this, might have to use nbhd. in \( xz \)-plane.

Easy example: Any function \( \phi(x) : \mathbb{R}^k \rightarrow \mathbb{R}^m \)

defines a \( k \)-dim'l manifold in \( \mathbb{R}^{k+m} \)

works for all points in \( \mathbb{R}^{k+m} \). Only true we expect to get global defi.

\[ x^2 + y^2 + z^2 - 1 = 0 \]

\( \text{unit sphere:} \)

\( \text{try this on own with} \)
e.g. \( f(x,y) = 0 : \mathbb{R}^2 \to \mathbb{R} \). \( f \) defines manifold structure on \( \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \subseteq \mathbb{R}^{2+1} = \mathbb{R}^3 \).

but this is just \( \mathbb{R}^2 \).

\( \mathbb{R}^2 \) is manifold.


\[ \textbf{Non-examples: } \text{Self-intersecting curves in the plane:} \]

\[ \text{is } X \text{ the graph of a function?} \]

No, fails one-one in both directions.

Another non-example: sharp point in plane:

\[ \text{is locally } \nabla \text{ in nbhd of sharp point.} \]

Not defined in open nbhd of \( y_0 \), not diff. in open nbhd of \( x_0 \).

Rigorous proof that any self-intersecting curve is not a manifold might be hard. But if we have equations, it is easier.

Simpler example: \( xy = 0 \).

\[ \text{graph is coord. axes.} \]

\[ \text{Fails one-one-ness locally at } (0,0). \]

If we remove origin, then we do have manifold.

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Implicit function theorem gives us

Many more examples of smooth manifolds:

Thm 3.1.10 in Hubbard: \( U \subseteq \mathbb{R}^n \) open, \( F : U \to \mathbb{R}^{n-k} \) a \( C^1 \) mapping

\[ M = \{ \bar{z} \in U \mid F(\bar{z}) = 0 \} \] "Zero locus of \( F \)"

If \( D_f(\bar{z}) \) is onto \( \forall \bar{z} \in M \), then \( M \) is a smooth \( k \)-dim manifold embedded in \( \mathbb{R}^n \) (proof is just implicit function theorem)
There is a converse: If \( M \) is a smooth \( k \)-dim manifold embedded in \( \mathbb{R}^n \), then every \( z \in M \) has nbhd \( U \subset \mathbb{R}^n \) s.t. \( \exists C^1 F: U \to \mathbb{R}^{n-k} \) with \( DF(z) \) onto and \( M \cap U = \{ y \in U \mid F(y) = 0 \} \).

Write \( z = \begin{bmatrix} y \\ x \end{bmatrix} \) indep. consider \( F(z) = x - f(y) = 0 \) (prove this Wednesday).

Nice remark in book: If \( DF(z) \) fails to be onto, doesn't mean that zero locus isn't smooth manifold. (E.g., dirty trick \( F(z)^2 = 0 \))

Example in book: \( x^4 + y^4 + x^2 - y^2 = c \).

For which \( c \) does this define smooth manifold? \( c = -1/4 \), \( c = 0 \)

fail to be onto

\( c = -1/4 \): two points. Is this a smooth manifold?

\( c = 0 \): 0-dimensional.

\( (0, \pm \sqrt{2}) \)

\( c = 0 \): figure eight curve.

\( k \)-dim manifold in \( \mathbb{R}^n \) is \( k \)-locally defined implicitly defined functions \( \left( \frac{x}{\phi(x)} \right) \) \( k \)-variables in \( \mathbb{R}^n \)

(reordered them here so they occur first in vector notation )