Last time, discussing smooth manifolds — hypersurfaces that locally are graph of differentiable function. (i.e. C^1 fn.)

Thm 3.1.10: Zero locus of $F : \mathbb{R}^{m+n} \to \mathbb{R}^m$, $F : \mathcal{C}^1$, $DF(\mathbf{z})$ onto, implicitly defines smooth manifold: $V + \mathbf{z}$ in locus $M := \{ \mathbf{z} \in U | F(\mathbf{z}) = 0 \}$ is smooth, n-dim'l manifold.

(pf: Implicit function theorem holds all $\mathbf{z} \in M$)

Example: $x^4 + y^4 + x^2 - y^2 = c$. For which $c$ does this map $\mathbb{R}^2 \to \mathbb{R}$ have assoc. zero locus defining smooth manifold? (1-dim'l) according to theorem.

Last time: Find where derivative fails to be onto for $(x,y) \in \mathbb{R}^2$.

See if these points lie on our zero locus.

$DF = \left[ \begin{array}{c} 4x^3 + 2x \\ 4y^3 - 2y \\ 2(2x^2 + 1) \\ 2y(y^2 - 1) \end{array} \right]$. Only place not onto if

$DF = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right]$

only = 0 if $x = 0$, $y = 0$ or $y = \pm 1/\sqrt{2}$

if $x = 0$, $y = 0$, $x^4 + y^4 + x^2 - y^2 = 0$. So $c = 0$

if $x = 0$, $y = \pm 1/\sqrt{2}$, $0 + 1/4 + 0 - 1/2 = -1/4$ so $c = -1/4$ (c = const. guarantee manifold structure)

$c = 0$ case: Figure 8 curve. Not manifold.

$c = -1/4$: two isolated points $(0, \pm 1/\sqrt{2})$. 0-dim'l manifold.
**Thm 3.1.10:** If \( M \) smooth, \( k \)-dim'l manifold in \( \mathbb{R}^{n+m} \), then every \( z \in M \) has nbhd \( U \subseteq \mathbb{R}^{n+m} \), \( C^1 \) function \( F : U \to \mathbb{R}^n \) s.t. \( DF(z) \) onto and \( M \cap U = \left\{ y \mid F(y) = 0 \right\} \).

**pf sketch:** Write \( z = \begin{bmatrix} y \\ x \end{bmatrix} \), \( y \) \( n \) indep. \( x \) \( m \) dep. Consider \( F(z) := x - f(y) \) zero locus.

This is onto \( \mathbb{R}^m \) since Jacobian is identity in columns corresponding to \( m \times 1 \) variables.

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**Thm:** (Inverse image of manifold is manifold)

\( M \subseteq \mathbb{R}^m \), \( k \)-dim'l manifold, with \( U \) open set in \( \mathbb{R}^n \), \( f : U \to \mathbb{R}^m \) with \( DF \) onto for all \( x \in f^{-1}(M) \). Then \( f^{-1}(M) \) in \( C^1(U) \) inverse image of \( f \), a set. \( \mathbb{R}^n \) of dim \( k+n-m \).

**Cor:** Let \( g \) be the map \( \mathbb{R}^n \to \mathbb{R}^n \)

\[ x \mapsto A \cdot x + \xi \]

then if \( M \) smooth \( k \)-manifold, with \( A \) invertible.

\( g(M) \) is smooth \( k \)-manifold.

**pf of Cor:** (assuming the theorem) \( g^{-1} = f \) where \( f : x \mapsto A^{-1}(x-\xi) \) and \( g(M) = f^{-1}(M) \). Now apply thm to \( f \). (In short, didn't like direct images so used inverse image of inverse map! really mean inverse.)
In other words, notion of manifold is coordinate-free since all changes of coordinates are of form \( x \rightarrow Ax + c \).

**Proof of Theorem:** Given \( a \in f^{-1}(M) \), know that since \( f(a) \in M \) is a nbhd \( V \) of \( f(a) \) s.t. points of \( M \cap V \) are given by zero locus \( F(y) = 0 \), \( F: V \rightarrow \mathbb{R}^{m-k} \), \( C^k \) mapping with \( DF \) onto for all \( x \in M \cap V \).

Since \( f \) continuous \( f^{-1}(V) \) open neighborhood of \( f(a) \), and the set \( f^{-1}(M) \cap f^{-1}(V) \) is the solution to \( F \circ f = 0 \).

Need to check \( D(F \circ f) \) onto. Do this by chain rule. \( \checkmark \) since \( DF(f(y)) \) onto by part 2 of earlier then, \( Df(y) \) onto by assumption in thm.

Book goes on long discussion about parametrizations versus implicitly defined functions, by check a pt. is on equation's zero locus, but don't know how to find pts on locus in first place.

Just the opposite is true for parametrizations.

Further trouble with parametrizations: hard to explain why they have manifold structure, if indeed they do.

(Key point: mapping must be one-one.

\[ \begin{align*}
e.g. \quad & \mathbb{R} \rightarrow M \\
& t \rightarrow \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}
\end{align*} \]

Locus of points traced out is manifold, hard to prove from parametrization, do in patches.)
Tangent spaces: Recall tangent hyperplane to a function \( f(x) \) at \( x_0 \),
\[
f: \mathbb{R}^k \to \mathbb{R}^m, \quad \text{is} \quad T(x) - f(x_0) = \left[ \frac{Df(x_0)}{1} \right] (x - x_0)
\]
with \( Df(x_0): \mathbb{R}^k \to \mathbb{R}^m \)
linear transformation.

We can make the same definition for a smooth \( k \)-manifold. By defin.
for any \( x_0 \in M \), locally the graph of \( C^1 \) function. For \( x \in B_\epsilon(x_0) \),
\[
\exists \; \tilde{x} = \left[ \begin{array}{c} x_1 \\ \vdots \\ x_k \\ y_1 \\ \vdots \\ y_k \end{array} \right] = \left[ \begin{array}{c} x_k \\ \vdots \\ x_k \end{array} \right] = \left[ \begin{array}{c} f_1(x_1, \ldots, x_k) \\ \vdots \\ f_{k-1}(x_1, \ldots, x_k) \end{array} \right] \frac{f(x)}{x} \quad \text{(not tangent line)}.
\]

Definition: The tangent space to manifold \( M \) at point \( x_0 \in M \), denoted
\( T_{x_0}(M) \), is the graph of the linear transformation \( Df(x_0) \)
with \( f \) as above.

E.g. \( y = f(x) \), then \( Df(x_0) = f'(x_0) \)
and graph is linear function \( y = f'(x_0) \cdot x \)

Nice from linear algebra point of view. Tangent space
is vector space, but it is kind to think about
it as "anchored" to point \( (x_0, f(x_0)) \in \mathbb{R}^m \).