Indeed linear transformations can be composed. So here's a natural question:

Given l_1, l_2 linear transformations with assoc. matrices A_1, A_2

And $l_1 \circ l_2$ with associated matrix B, can we give matrix operation

$*$ so that $A_1 * A_2 = B$? i.e. what operation on matrices corresponds to composition?

Answer, of course, is matrix multiplization and explains our funny definition of matrix mult. in the first place.

But let's prove it carefully: $l_2 : \mathbb{R}^n \to \mathbb{R}^m \quad l_1 : \mathbb{R}^m \to \mathbb{R}^k$

Part 1: If l_1, l_2 linear, then $l_1 \circ l_2$ linear

$$l_1 \circ l_2 (c_1 \vec{u} + c_2 \vec{v}) = l_1 (c_1 l_2(\vec{u}) + c_2 l_2(\vec{v})) \quad \text{since } l_2 \text{ lin.}$$

$$= c_1 \, l_1 \circ l_2 (\vec{u}) + c_2 \, l_1 \circ l_2 (\vec{v}) \quad \text{since } l_1 \text{ lin.} \checkmark$$

Part 2: To find matrix of $l_1 \circ l_2$, evaluate standard basis vectors $\vec{e}_i$.

$$l_1 \circ l_2 (\vec{e}_i) = l_1 (\text{ ith column vector of } A_2)$$

call it

$$\begin{bmatrix} a^{(2)}_{1,i} \\ \vdots \\ a^{(2)}_{m,i} \end{bmatrix}$$

Now it gets a little messy: Write this as

(*) $a^{(2)}_{1,i} \vec{e}_1 + \cdots + a^{(2)}_{m,i} \vec{e}_m$ and note $l_1(\vec{e}_j) = j^{th}$ column of A_1

$$\text{so } (*) = \begin{bmatrix} a^{(1)}_{1,1} a^{(2)}_{1,i} + \cdots + a^{(1)}_{1,m} a^{(2)}_{m,i} \\ \vdots \\ \\ \end{bmatrix} \qquad = \begin{bmatrix} a^{(1)}_{1,j} \\ \vdots \\ a^{(1)}_{k,j} \end{bmatrix}$$

Now that we know all linear transformations $l: \mathbb{R}^n \to \mathbb{R}^m$ are associated with $m \times n$ matrix, ask questions about matrix mult. and answer with linear trans. knowledge.

<u>Q1</u> : Is matrix multiplication associative? Yes.

Proof 1: check it directly from defn

Proof 2: composition of functions is associative.

<u>Q2</u> : Is matrix multiplication commutative?

Not in general. Find two linear transformations that don't commute.

Do example of reflection $\circ$ rotation.

But some matrices do commute – think of examples – e.g. pair of rotations.

<u>Q3</u> : When does a linear transformation have an inverse?

Given l, seek linear transformation l' s.t. $l \circ l' = Id_{\mathbb{R}^m \to \mathbb{R}^m}$

$l: \mathbb{R}^n \to \mathbb{R}^m$ $\quad l': \mathbb{R}^m \to \mathbb{R}^n$ $\quad l' \circ l = Id_{\mathbb{R}^n \to \mathbb{R}^n}$

<u>Non-examples</u> : What about projection to a line?

No, since function is not onto and not one-one.

In fact, Hubbard and Hubbard define a linear transformation to be <u>invertible</u> if <u>one-one</u> and <u>onto</u>

one-one: unique image for each point in $\mathbb{R}^n$

onto: hit every point in $\mathbb{R}^m$

(this is equivalent to definition above) as these conditions ensure well-defined inverse

often these conditions easier to check in practice

You might be thinking that $f: \mathbb{R}^n \to \mathbb{R}^m$ can only be one-one, onto if $m=n$. i.e. if $m>n$, can't map to all points.

e.g. $f: \mathbb{R} \to \mathbb{R}^2$

or if $f: \mathbb{R}^2 \to \mathbb{R}$ then "not enough room". Some points in $\mathbb{R}^2$ must go to same place.

Sneaky point: True that only linear functions $l: \mathbb{R}^n \to \mathbb{R}^m$ that are invertible have $m=n$.

However, there are onto functions $f: \mathbb{R} \to \mathbb{R}^2$ "Peano curves", but the f is not linear (nor continuous)

Proposition: (1.3.14 in Hubbard and Hubbard) Let l be linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ with matrix A

l is invertible if and only if

$$\exists \ B \text{ s.t. } A \cdot B = B \cdot A = I$$

identity transformation as matrix: $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

Better to write.

$$A \cdot B = I_m, \quad B \cdot A = I_n.$$

we say B is a "matrix inverse" for A. write $B = A^{-1}$.

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pf: Suppose A invertible with inverse B. Show that linear map assoc. to B, call it l', is the inverse of l.

Given any $\vec{y} \in \mathbb{R}^m$ then $\vec{y} = I_m \cdot \vec{y} = A \cdot B \cdot \vec{y} = \underset{\substack{\parallel \\ l}}{l_A} \cdot \underset{\substack{\parallel \\ l'}}{l_B} \cdot \vec{y}$ shows l onto

Given any $\vec{x} \in \mathbb{R}^n$ then $\vec{x} = I_n \vec{x} = B \cdot A \vec{x} = l' \cdot l \vec{x}$

shows l one-one

Other direction: If l invertible with inverse l' and assoc. matrix B,

then since $l \circ l' = Id_{\mathbb{R}^n \to \mathbb{R}^n}$, then $A \cdot B = I_n$ since $A \cdot B$ is matrix assoc. to $l \circ l'$

Similarly $l' \circ l = Id_{\mathbb{R}^m \to \mathbb{R}^m}$

$\Rightarrow B \cdot A = I_m$.

One problem: Haven't showed l' linear yet, so can't assume l' is assoc. to a matrix B.

i.e. show if l linear with inverse l', then

for any $\vec{y}_1, \vec{y}_2 \in \mathbb{R}^m$, $c_1, c_2 \in \mathbb{R}$: $l'(c_1\vec{y}_1 + c_2\vec{y}_2)$

$= c_1\, l'(\vec{y}_1) + c_2\, l'(\vec{y}_2)$.

pf: $l \circ l'(c_1\vec{y}_1 + c_2\vec{y}_2) = c_1\vec{y}_1 + c_2\vec{y}_2$

$= c_1\, l \circ l'(\vec{y}_1) + c_2\, l \circ l'(\vec{y}_2)$

$= l(c_1\, l'(\vec{y}_1) + c_2\, l'(\vec{y}_2))$

since l linear

Now use that l one-one, so if $l(\vec{v}) = l(\vec{u})$ then $\vec{v} = \vec{u}$.

i.e. in our case $l'(c_1\vec{y}_1 + c_2\vec{y}_2) = c_1\, l'(\vec{y}_1) + c_2\, l'(\vec{y}_2)$ as desired.