Indeed linear transformations can be composed. So here's a natural question:

Given \( l_1, l_2 \) linear transformations with associated matrices \( A_1, A_2 \).

Given \( l_1 \circ l_2 \) with associated matrix \( B \), can we give matrix operation \( \ast \) so that \( A_1 \ast A_2 = B \)? i.e. what operation on matrices corresponds to composition?

Answer, of course, is matrix multiplication and explains our funny definition of matrix mult. in the first place.

But let's prove it carefully:

\[ l_2 : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad l_1 : \mathbb{R}^m \rightarrow \mathbb{R}^k \]

**Part 1:** If \( l_1, l_2 \) linear, then \( l_1 \circ l_2 \) linear

\[
l_1 \circ l_2 \left( c_1 \bar{u} + c_2 \bar{v} \right) = l_1 \left( c_1 l_2(\bar{u}) + c_2 l_2(\bar{v}) \right) \quad \text{since } l_2 \text{ lin.}
= c_1 l_1 \circ l_2(\bar{u}) + c_2 l_1 \circ l_2(\bar{v}) \quad \text{since } l_1 \text{ lin.}
\]

**Part 2:** To find matrix of \( l_1 \circ l_2 \), evaluate standard basis vectors \( \bar{e}_i \).

\[
l_1 \circ l_2 \left( \bar{e}_i \right) = l_1 \left( \text{ith column vector of } A_2 \right)
\]

Now it gets a little messy. Write this as

\[
\begin{bmatrix}
a_{11}^{(2)} \\
\vdots \\
a_{m1}^{(2)}
\end{bmatrix}
\]

\[
\begin{bmatrix}
a_{11}^{(1)} \\
\vdots \\
a_{m1}^{(1)}
\end{bmatrix}
\]

So (k) =

\[
\begin{bmatrix}
a_{11}^{(1)} a_{11}^{(2)} + \ldots + a_{1m}^{(1)} a_{m1}^{(2)} \\
\vdots \\
a_{11}^{(1)} a_{11}^{(2)} + \ldots + a_{1m}^{(1)} a_{m1}^{(2)}
\end{bmatrix}
\]

And note \( l_1 \left( \bar{e}_j \right) = j^{th} \text{ column of } A_1 \).
Now that we know all linear transformations \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) are associated with \( m \times n \) matrix, ask questions about matrix mult., and answer with linear trans. knowledge.

Q1: Is matrix multiplication associative?  

Proof 1: check it directly from definition.  

Proof 2: composition of functions is associative.

Q2: Is matrix multiplication commutative?  

Not in general. Find two linear transformations that don't commute.  

Do example of reflection \( \circ \) rotation.

But some matrices do commute — think of examples — e.g., pair of rotations.

Q3: When does a linear transformation have an inverse?  

Given \( L \) seek \( L' \) s.t. \( L \circ L' = \text{Id}_{\mathbb{R}^n \rightarrow \mathbb{R}^n} \)

\( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad L' : \mathbb{R}^m \rightarrow \mathbb{R}^n \)

Non-examples: What about projection to a line?  

No, since function is not onto and not one-one.

In fact, Hubbard and Hubbard define a linear transformation to be invertible if one-one and onto — hit every point in \( \mathbb{R}^m \) (this is equivalent to definition above) as these conditions ensure well-defined inverse.
You might be thinking that \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) can only be one-one, onto if \( m = n \), i.e., if \( m > n \), can't map to all points.

Example: \( f: \mathbb{R} \rightarrow \mathbb{R}^2 \)

or if \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) then "not enough room". Some points in \( \mathbb{R}^2 \) must go to same place.

Supplementary point: True that only linear functions

\( l: \mathbb{R}^n \rightarrow \mathbb{R}^m \) that are invertible

However, there are onto functions \( f: \mathbb{R} \rightarrow \mathbb{R}^2 \)

"Fano curves", but the \( f \) is not linear (nor continuous)

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Proposition: (1.3.14 in Hubbard and Hubbard) Let \( l \) be linear transformation with matrix \( A \)

\( l \) is invertible if and only if

\[ AB = BA = I \]

there exist \( B \) s.t. \( A \cdot B = B \cdot A = I \)

Identify transformation as matrix:

\[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \]

we say \( B \) is a "matrix inverse" for \( A \), write \( B = A^{-1} \).

Better to write:

\[ A \cdot B = I_m \quad B \cdot A = I_n \]

---

Proof: Suppose \( A \) invertible with inverse \( B \). Show that linear map associates with \( B \).

To \( B \), call it \( l' \)'s the inverse of \( l \).

If \( y \in \mathbb{R}^n \) then \( \bar{y} = A \cdot \bar{y} \) = \( A \cdot B \cdot \bar{y} = l' \cdot \bar{y} \)

Given \( y \in \mathbb{R}^n \) then \( z = A \cdot \bar{y} \)

shows \( l' \) one-one

---

If \( x \in \mathbb{R}^n \) then \( \bar{x} = B \cdot \bar{y} \) = \( l' \cdot \bar{x} \)

shows \( l' \) onto
Other direction: If $l$ invertible with inverse $l'$ and assoc. matrix $B$, then since $l \circ l' = \text{Id}_{\mathbb{R}^m \rightarrow \mathbb{R}^n}$, then $A \cdot B = I_n$

Similarly $l' \circ l = \text{Id}_{\mathbb{R}^m \rightarrow \mathbb{R}^n}$ since $A \cdot B$ is matrix assoc. to $l \circ l'$

$\Rightarrow B \cdot A = I_m$.

One problem: Haven't showed $l'$ linear yet, so can't assume $l'$ is assoc. to a matrix $B$.

i.e. show if $l$ linear with inverse $l'$, then

for any $\vec{y}_1, \vec{y}_2 \in \mathbb{R}^m$, $c_1, c_2 \in \mathbb{R}$:

$l' (c_1 \vec{y}_1 + c_2 \vec{y}_2) = c_1 l'(\vec{y}_1) + c_2 l'(\vec{y}_2) = c_1 l'(\vec{y}_1) + c_2 l'(\vec{y}_2)$.

Hence:

$l \circ l' (c_1 \vec{y}_1 + c_2 \vec{y}_2) = c_1 l'(\vec{y}_1) + c_2 l'(\vec{y}_2)$

$= c_1 l \circ l' (\vec{y}_1) + c_2 l \circ l' (\vec{y}_2)$

$= l (c_1 l'(\vec{y}_1) + c_2 l'(\vec{y}_2))$

since $l$ linear

Now use that $l$ one-one, so if $l (\vec{v}) = l (\vec{u})$ then $\vec{v} = \vec{u}$.

i.e. $l' (c_1 \vec{y}_1 + c_2 \vec{y}_2) = c_1 l'(\vec{y}_1) + c_2 l'(\vec{y}_2)$ as desired.