Last week, we discussed tangent hyperplanes — locally approximate a manifold, itself locally graph of diff. function (in fact C^1)

And tangent space to a point, only interesting information in tangent hyperplane, is easily computed.

Given zero locus $F=0$ compute $\mathrm{Ker}\,[DF(\underline{z}_0)]$

Given parametrization $\gamma: U \to M$ compute $\mathrm{Im}\,[D\gamma(u)]$

Next natural question:

How good is local approximation? Give some quantitative measure of accuracy.

(Taylor's theorem with remainder).

Begin with discussion of Taylor polynomials in several variables.

In one-variable, Taylor polynomial to f at $x=a$ (of deg. k) is the polynomial whose first k derivatives match those of f at $x=a$:

$$p^k_{f,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Nice way to formulate precisely our intuitive notion that $p^k_{f,a}$ is close to f near $x=a$:

$$\lim_{h\to 0} \frac{f(a+h) - p^k_{f,a}(a+h)}{h^k} = 0.$$

(sketch pf. of more general result later...)

How should we generalize to several variables? $p^1_{f,\underline{a}}$: tangent hyperplane

$$[Df(\underline{a})](\underline{x}-\underline{a}) = D_1 f(\underline{a})\cdot(x_1-a_1) + \dots + D_n f(\underline{a})\cdot(x_n-a_n)$$

so first partials of tangent hyperplane at $\underline{x}=\underline{a}$ match those of f.

More generally, ask for polynomial to match partials of higher degree.

so $P^2_{f,\underline{a}}$ would match all partials up to degree 2 —

all combinations $D_i(D_j(f))$ with $i,j \in [1,\dots,n]$ (n^2 derivatives)

A lot of book keeping. Use multi-indices.

$D^{(e_1,\dots,e_n)}$ means $D_1^{e_1} \cdots D_n^{e_n}$, so $D^{(3,0,1)}$ means $D_1^3 \cdot D_2$. (ordered)

problem: No way to write $D_2 D_1^3$.

So this would be really not useful notation except that...

Thm: for $f \in D^2$ (twice diff.), mixed partials are equal!

for any i,j: $D_j \{ D_i (f(\underline{a}) \} = D_i(D_j f)(\underline{a})$ for any $\underline{a} \in \mathcal{U}$ with $f: \mathcal{U} \to \mathbb{R}^m$ in $D^2(\mathcal{U})$

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so $D_2 \circ D_1^3 = D_2 \circ D_1 \circ D_1 \circ D_1$
$= D_1 \circ D_2 \circ D_1 \circ D_1 = \cdots = D_1^3 \cdot D_2$. We can always order derivatives in this way.

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pf. of thm: check this component by component for $f: \mathcal{U} \to \mathbb{R}^m$,

so assume $m = 1$.

Write out $D_j(D_i f)(\underline{a}) := D_j \left(\lim_{s \to 0} \frac{f(\underline{x} + s\vec{e}_i) - f(\underline{x})}{s} \right) \Bigg|_{\underline{x} = \underline{a}}$

$$= \lim_{t \to 0} \lim_{s \to 0} \frac{f(\underline{a} + s\vec{e}_i + t\vec{e}_j) - f(\underline{a} + t\vec{e}_j) - f(\underline{a} + s\vec{e}_i) + f(\underline{a})}{s \cdot t} \quad (*)$$

if we could set $s = t$ then done, then just one limit whose expression is symmetric when we reverse roles of i and j.

strategy is to regroup terms; familiar trick in product rule, etc.,

justify the above by ~~[illegible]~~ Mean Value Theorem.

$$g_t(s) := f(\underline{a} + s\vec{e}_i + t\vec{e}_j) - f(\underline{a} + s\vec{e}_i).$$

Then $g_t(s) - g_t(0)$ is numerator in $(*)$, want to show the limit in $(*)$ matches limit in $t \to 0$ with $\dfrac{g_t(t) - g_t(0)}{t}$ replacing $\dfrac{g_t(s) - g_t(0)}{s}$.

For this MVT $\Rightarrow \exists\, c \in (0,t)$ such that this equals $g'(c)$. (g is diff. since it is made from f diff.)

call it c_t to emphasize its dependence on t.

In other words, we are reduced to showing

$$D_j(D_i f)(\underline{a}) = \lim_{t \to 0} \frac{g'(c_t)}{t}$$

First compute $g_t'(c_t)$ in terms of f:

$$= D_i f(\underline{a} + c_t\vec{e}_i + t\vec{e}_j) - D_i f(\underline{a} + c_t\vec{e}_i)$$

$D_i f$ is differentiable. From definition, approach $\underline{a}$ from $\underline{a} + c_t\vec{e}_i + t\vec{e}_j$:

$$0 = \lim_{(c_t,t) \to \underline{0}} \frac{1}{|(c_t,t)|} \Big(D_i(f(\underline{a} + c_t\vec{e}_i + t\vec{e}_j) - D_i f(\underline{a}) - \underbrace{[DD_i f(\underline{a})]}_{1 \times n \text{ matrix}}(c_t\vec{e}_i + t\vec{e}_j) \Big)$$

since $c_t < t$, can replace $|(c_t,t)|$ with $|(t,t)|$, conclude limit is 0 with $\frac{1}{|t|}$.

\# that records mult. in ith, jth component only, so get

$$c_t \cdot D_i^2 f(\underline{a}) + t\, \underbrace{D_j D_i f(\underline{a})}_{\text{second mixed partial we wanted}}.$$

Now approach $\underline{a}$ from $\underline{a} + c_t \vec{e}_i$ with limit as $t \to 0$. (so $c_t \to 0$)

$$0 = \lim_{t \to 0} \frac{1}{|c_t|} \left(D_i f(\underline{a} + c e_i) - D_i f(\underline{a}) - \underbrace{[D \cdot D_i f(a)] c \vec{e}_i}_{c D_i^2 \cdot f(a)} \right)$$

again $c_t < t$, so replace with $|t|$.

Compare these two:

$$\lim_{t \to 0} \frac{g'(c_t)}{t} = D_j D_i f(\underline{a}) \text{ as desired.}$$

Now have essential fact about mixed partials AND our multi-index is well defined.

Then define:

$$P^k_{f,\underline{a}}(\underline{x}) = \sum_{m=0}^{k} \sum_{(l_1, \dots, l_n) \in I_n^m} \frac{1}{l_1! \cdots l_n!} D^{(l_1, \dots, l_n)} f(\underline{a}) \cdot (\underline{x} - \underline{a})^{(l_1, \dots, l_n)}$$

clear that its partial derivs of order $\leq k$ will match those of $f(\underline{x})$ at $\underline{x} = \underline{a}$.

at $\underline{x} = \underline{a}$