Last week — Computing Taylor polynomials, for function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \).

Two facts: (1) partial derivatives required are given by multi-indices.

Multi-indices remind us that mixed partials equal, remind us of factorial, exponent attached to monomial.

(2) Shortcuts from composition/mult. rules for Taylor polynomials (power of little \( o / \mathcal{O} \) notation)

When we left off, working on Taylor polynomial to manifold at point \( \phi(y) \) (using its implicit function theorem)

\[
M \mapsto f(x) = x^3 + xy + y^3 - 3 = 0 \quad \text{at} \quad (x) = (1).
\]

\[
\frac{\partial f}{\partial y} = [4, 4] \Rightarrow \text{write } f = 0 \text{ locally as } (\phi(y)) \text{ (or in terms of } x) \text{ but hard to define } \phi \text{ explicitly in general,}
\]

Options: (1) Harder: use higher derivatives + chain rule, to find expressions for partial derms of \( \phi \).

(2) Better: Use

\[
\left[ P_{F_i} \left( \frac{\partial}{\partial b} \right) \right]^k \left( \frac{\partial \phi}{\partial b} \right) \left( b + h \right) = 0 \quad \text{up to order } k.
\]

return to example from last Wednesday —

Final topic — Taylor’s theorem with remainder, estimates difference between \( f, \phi \).

In one variable

\[
f(\alpha + h) = f(\alpha) + \int_{\alpha}^{\alpha + h} f'(t) \, dt \quad \text{(fundamental thm. of calc.)}
\]

Repeat using integration by parts:

\[
f(\alpha + h) = f(\alpha) + f'(\alpha) \left( h + \frac{h}{2!} \right) + \ldots + \frac{f^{(k)}(\alpha)}{k!} h^k + \\
+ \frac{1}{k!} \int_0^h (\alpha + k + t) f^{(k+1)}(h + t) \, dt
\]
Can't evaluate this latter integral, but we can estimate it using

Mean Value Theorem: \[ \exists c \in (a, a + h) \text{ s.t.} \]

\[
\frac{1}{k!} \int_0^h (h-t)^k f^{(k+1)}(a + t) \, dt = f^{(k+1)}(c) \cdot \frac{1}{k!} \int_0^h (h-t)^k \, dt
\]

Don't know where c is exactly in \((a, a + h)\),

but if we bound \(f^{(k+1)}(x)\) on this interval,

\((\text{say } |f^{(k+1)}(x)| \leq C\text{, some const.} C\) on this interval),

then

\[
|\text{ERROR}| = \left| f(a + h) - P_{f,a}^k(a + h) \right| \leq \frac{C}{(k+1)!} h^{k+1}
\]

Example: \(e^{-x^2}\) at \(x = 0\). \(x \rightarrow -x^2\) so can compose \(x \rightarrow -x^2\)

with Taylor expansion for \(e^x\) at \(x = 0\)

\(e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2} + o(x^4)\)

What is bound for error of \(e^{-1/4}\) using \(P_{f,0}^2 = 1 - x^2\)?

We estimate at \(x = 1/2\) that \(e^{-1/4} \approx 3/4 \approx 0.75\)

\[
|\text{ERROR}| \leq \frac{C}{3!} \left( \frac{1}{2} \right)^3 = \frac{C}{48}, \text{where } |f^{(3)}(x)| \leq C \text{ on } (0, 1/2)
\]

\[f(x) = e^{-x^2}\]
\[f'(x) = -2x e^{-x^2}\]
\[f''(x) = 4x^2 e^{-x^2} - 2e^{-x^2}\]
\[f'''(x) = -8x^3 e^{-x^2} + 8xe^{-x^2} + 4xe^{-x^2} = -8x^3 e^{-x^2} + 12x e^{-x^2} - 4x(x^2 - 3)e^{-x^2}\]
A similar result exists for multi-var. functions: ( clever application of one var. result

\[ \exists c \in [a, a+h] \text{ s.t.:} \]

\[ f(a+h) - P^k_{f,c}(a+h) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{D_i f(c)}{!} h^i \]

So need to bound all these with \( c \in [a, a+h] \).
Say by constant \( C \)

\[ |\text{Error}| \leq C \left( \sum_{i=1}^{n} |h_i| \right)^k \quad \text{need } f \in C^{k+1} \]

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Next topic: classifying local max/min of functions

In one variable, used second derivative test: if \( f'(a) = 0 \) and...

If \( f''(a) > 0 \): min, if \( f''(a) < 0 \): max.

Recast: quadratic term in Taylor expansion is pos./neg.

What about functions \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \).

\( (x,y) \mapsto f(x,y) \)

Top terms in quadratic Taylor polynomial have form \( a_{(2,0)} x^2 + a_{(1,1)} xy + a_{(0,2)} y^2 \).

How to understand behavior of terms like this?

Prototypes:

\[ f(x,y) = x^2 + y^2 \quad \text{"paraboloid"} \]

\[ f(x,y) = -x^2 - y^2 \quad \text{same, but upside down} \]

\[ f(x,y) = x^2 - y^2 \quad \text{or} \quad y^2 - x^2 \quad \text{"Saddle"} \]