Local maxima/minima of diff. functions.

In one-variable \( f : U \subseteq \mathbb{R} \to \mathbb{R} \), we find critical points:

(places where \( f \) not diff., e.g., sharp point — exclude those for now and just discuss \( f \) diff. on all of \( U \).)

Further test to decide if \( f \) has local max/min —

Second derivative test: if \( f'(a) = 0, f''(a) > 0 \) \( \Rightarrow \) local min at \( a \)

if \( f'(a) = 0, f''(a) < 0 \) \( \Rightarrow \) local max at \( a \)

if \( f'(a) = 0, f''(a) = 0 \) \( \Rightarrow \) inconclusive.

E.g., \( f(x) = \begin{cases} x^4 & \text{if } x > 0, \\ x^3 & \text{if } x < 0 \end{cases} \) is neither.

Same plan for \( f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m \), diff. on \( U \)

Say \( f \) has critical point at \( a \in \mathbb{R}^n \) if \( \begin{bmatrix} \frac{\partial f(a)}{\partial x_1} \\ \vdots \\ \frac{\partial f(a)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \) w. the \( 1 \times n \) 0-matrix.

i.e., tangent hyperplane to graph is parallel to \( \mathbb{R}^n \) hyperplane.

picture: \( f : \mathbb{R}^2 \to \mathbb{R} \) with \( \begin{bmatrix} \frac{\partial f(a)}{\partial x_1} \\ \frac{\partial f(a)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

then tangent plane has equation \( z = f(a) \), i.e., a horizontal plane, parallel to \( xy \)-plane.

Remarks:

1. Solve for which \( a \in \mathbb{R}^n \) have
   \( \begin{bmatrix} \frac{\partial f(a)}{\partial x_1} \\ \vdots \\ \frac{\partial f(a)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \)
   using Newton's method.

2. What about second derivative test?
Rephrase second derivative test as asking about quadratic coeff. in
Taylor expansion of $f$.

What is analogue of quadratic term in multi-variable expansion?

All terms whose total degree is 2.

Ex. $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $p^2_{f|a}$ has degree two terms $a_{2,0} x^2 + a_{1,1} xy$

Easy examples: $x^2 + y^2 = Q(x,y)$

```
Graph this by looking at slices, like

$x = 0$, $y = 0$, $y = x$
```

So if $p^2_{f|a}$ had top terms of form $x^2 + y^2$, suspect that $f$ has local min at $a$.

Similarly $-x^2 - y^2 = Q(x,y)$ flips graph over $x$-$y$ plane.

suggests $f$ would have local max at $a$ if these were top terms in $p^2_{f|a}$.

What about $Q(x,y) = x^2 - y^2$?

Graph a little hard to picture. Again think of slices.

```
Move in $x$ direction increase, in $y$ direction decrease so this is neither max nor min.
```

"saddle"
What about arbitrary looking top form like

\[ x^2 + 2xy + y^2 \]

Write as \((x+y)^2 \). (> 0, suggests if
this appeared in
\[ p^2 \]
then it \( \rightarrow \), that a
is local min.)

Plan: Complete the square.

\[ x^2 + xy + \left( \frac{y}{2} \right)^2 = \left( x + \frac{y}{2} \right)^2 \]

so \[ x^2 + xy + y^2 = x^2 + xy + \left( \frac{y}{2} \right)^2 + \frac{3y^2}{4} \]

\[ = \left( x + \frac{y}{2} \right)^2 + \left( \frac{\sqrt{3}y}{2} \right)^2 \]

whereas \[ x^2 + xy - y^2 = \left( x + \frac{y}{2} \right)^2 - \left( \frac{\sqrt{3}y}{2} \right)^2 \]

saddle in coords \( x + y/2 \)

could we have done it differently? No.

Theorem: Given quadratic form \( Q : \mathbb{R}^n \rightarrow \mathbb{R} \), \( F \) \( m = k + l \)
linearly indep. functions \( \alpha_1, \ldots, \alpha_m : \mathbb{R}^n \rightarrow \mathbb{R} \) s.t.

\[ Q(x) = \alpha_1(x)^2 + \ldots + \alpha_k(x)^2 - \alpha_{k+1}(x)^2 - \ldots - \alpha_{k+l}(x)^2 \]

And # plus signs \( = k \) is independent of choice of \( \alpha_i \);

# minus signs \( = l \) (intrinsic to \( Q \)).

Call the pair \((k, l)\) the "signature" of \( Q \). (Better: "footprint")

linearly independent linear functions \( \alpha_1, \ldots, \alpha_m \):

\[ \text{identically as functions, } \int \alpha_i(x)^2 \text{ for all values } x \]
if \( \] then \( \alpha_1(x)^2 + \ldots + \alpha_m(x)^2 = 0 \) Check this by finding out if
\[ c_1, \ldots, c_m \text{ with } c_1\alpha_1 + \ldots + c_m\alpha_m = 0 \]
then \( c_1 = 0, \ldots, c_m = 0 \). Check this by finding out if
\[ \text{nxm matrix made from } \alpha_i \text{'s has full rank.} \]
prove that such form is possible by induction.

1 variable quadratic form is \( c \cdot x^2 \), \( c \in \mathbb{R} \). \( \checkmark \).

For \( n \) variable form, two cases:

(a) \( \exists \) term of form \( c_i \cdot x_i^2 \) for some \( c_i \in \mathbb{R} \), \( i \in \{1, \ldots, n\} \).

then gather terms with \( x_i \) and complete square.

What's left is in \( n-1 \) vars, so use induction hypothesis.

Easy to check resulting linear functions are independent by evaluating at \( c_i \).

Example: \( Q(\mathbf{x}) = x_1^2 + 2x_1x_2 + 6x_1x_3 - x_3^2 \)

Pick \( x_1 \): \( x_1^2 + (2x_2 + 6x_3) \cdot x_1 \rightarrow \) complete the square:

\[
\left( x_1 + x_2 + 3x_3 \right)^2 - \left( x_2 + 3x_3 \right)^2 - x_3^2. \checkmark
\]

(b) No square terms in \( Q \). Just mixed quadratic terms \( c_{ij} x_i x_j \).

Do initial substitution \( x_i \mapsto x_j + u \) with \( i \neq j \).

if for some non-zero \( c_{ij} x_i x_j \) is in \( Q \),

this monomial becomes \( c_{ij} (x_j^2 + ux_j) \). Revert to previous case, now with vars \( x_1, \ldots, x_n, u \) not \( x_i \).