Examples: \( z^2 = x^2 + y^2 \) cut by plane \( z = 1 + x + y \).

Call result \( C \). Find closest point to origin on \( C \).

Let \( D \) be domain bounded between \( x + y = 0 \) and \( x^2 + y^2 = 1 \).

Maximize \( f(x, y) = xy \) on \( D \) or minimize

\[
\begin{align*}
\min & \quad f(x, y) = 5x + 5xy \\
\text{subject to } & \quad x^2 + y^2 = 1 \\
\end{align*}
\]

Critical points of \( f(x, y, z) = xyz \) on surface \( x + y + z^2 = 16 \). Is there a maximum?

Find max of \( x_1, \ldots, x_n \) subject to constraint \( x_1^2 + 2x_2^2 + \ldots + nx_n^2 = 1 \).

\[
\begin{align*}
f(x, y) &= xy + z^2 \\
\text{subject to } & \quad x^2 + y^2 + (z - 1)^2 = 1 \\
\end{align*}
\]

Find shortest distance between ellipse \( x^2 + 2y^2 = 2 \) and line \( x + y = 2 \). 

\( f \) : square of distance, 
Constraint: \( \mathbb{R}^4 \to \mathbb{R}^2 \) on both curves.
One theoretical application of Lagrange multipliers is the spectral theorem.

Remember that an eigenvector $\tilde{v}$ for transformation $A$ is a vector for which $\exists \lambda$ s.t. $A\tilde{v} = \lambda \tilde{v}$ (i.e. $A$ acts by scaling).

$A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (linear)

If we can find $v_1, \ldots, v_n$ a basis of $\mathbb{R}^n$ with $v_i$ eigenvectors.

$Q$: change of basis matrix from $e_i$ to $v_i$ then $Q^{-1}AQ$ is diagonal matrix $\begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix}$ of eigenvalues.

Big question: when can this be done?

Not always. E.g. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is rotation by 90°. No non-zero vector is scaled under such a rotation (or any other rotation $\neq 180°$).

Spectral Theorem: If $A$ is a symmetric, real $n \times n$ matrix, $(A = A^T)$ then $E$ has a basis of eigenvectors $v_1, \ldots, v_n$ (in fact, chosen to be orthonormal) of size 1, mutually orthogonal.

Proof uses Lagrange multipliers. Rough idea:

$A$: symmetric $\leftrightarrow \text{Quadratic form } QA$ where $QA(z) = (z_1 \ldots z_n) \cdot A \cdot (z_1 \ldots z_n)^T$

$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \rightarrow QA(z_1, z_2) = \begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

bijection because $QA$ has 3 coefficients, $A$ has 3 distinct entries.
Infer properties of \( A \) from those of \( Q_A \), viewed as map \( \mathbb{R}^n \rightarrow \mathbb{R} \), where we impose additional constraints.

Just do beginning: \( Q_A \) restricted to unit sphere \( \sqrt{\sum x_i^2} = 1 \).

\[ S \]

\[ \text{compact set, } \ F: 1 \cdot \sum x_i^2 - 1 = 0 \]

Compute

\[ [DQ_A(c)] \cdot [DF(c)] \]

so \( Q_A \mid_S \) has max/min.

More elegant to write them as linear transformations:

\[ [DF(c)] \cdot h = 2 \ c \cdot \ h \quad \text{or} \quad 2 \ c^T \ h. \]

\[ [DQ_A(c)] \cdot h = c \cdot Ah + h \cdot Ac \]

\[ = 2 \ c^T \ A \ h \]

Maximum has \( \lambda_i \) such that

\[ 2 \ c^T \ A = \lambda_i \ 2 \ c^T \]

\[ \Rightarrow A^T \ c = \lambda_i \ c \]

so \( c \) is eigenvector with length 1.

Assume symmetric.

Next constraint: lie on

\( S : \text{sphere} \) with \( x \cdot v_1 = 0 \)

orthogonal.

Cell \( \mathbb{H} \ v_1 \).