Open sets in \( \mathbb{R}^n \). (Geometric intuition better if we think of points rather than vectors.)

First, open ball \( B_r(x) = \{ y \in \mathbb{R}^n | \| x - y \| < r \} \).

Then open set of \( \mathbb{R}^n \): \( U \) is open in \( \mathbb{R}^n \) if, for every \( x \in U \), there is some \( r \) such that \( B_r(x) \subseteq U \).

Discuss examples:

1. Open interval \((a, b)\) is open in \( \mathbb{R} \). Warning: not open in \( \mathbb{R}^2 \) since ball is taken in given ambient space.

   Any open ball is open set.

   "Shapes" like square w/o boundary. Do some non-examples.

We want functions to be defined on open sets, so that neighborhoods exist for any point, and we may approach in any direction.

Closed set: complement of open set. So \( C \subseteq \mathbb{R}^n \) closed if \( \mathbb{R}^n - C \) is open.

We can also define closure/interior of set:

Idea: Given \( U \subseteq \mathbb{R}^n \), find smallest closed subset containing \( U \). largest open subset contained in \( U \).
make definitions (and in turn, check that the aforementioned characterizations are true)

\[ \text{Closure: } \text{Given } U \subseteq \mathbb{R}^n, \quad \overline{U} := \{ x \in \mathbb{R}^n \mid \exists r > 0 \text{ such that } B_r(x) \cap U \neq \emptyset \} \]

\[ \text{Interior: } \text{Given } U \subseteq \mathbb{R}^n, \quad \text{int } U := \{ x \in \mathbb{R}^n \mid \forall \epsilon > 0, \exists r > 0 \text{ such that } B_r(x) \subseteq U \} \]

In either case (again HW), you can characterize the points added/subtracted:

\[ \text{boundary: } \text{Given } U \subseteq \mathbb{R}^n, \quad \partial U := \text{boundary of } U = \{ x \in \mathbb{R}^n \mid \forall \epsilon > 0, \exists \text{ non-empty intersection of } x \text{ with } A, \mathbb{R}^n - A \} \]

Then

\[ \overline{U} = U \cup \partial U, \quad \text{int } U = U - \partial U. \]

so \( \partial U = \overline{U} - \text{int } U. \)

Do more examples... open unit disk - origin, \( |y| < x^2 \); between two parabolas

limits: limit of sequence (book rightly points out that need to get order of quantifiers straight)

\[ \xi a \in \mathbb{R}^n \]

converges to a point \( a \) if, for every \( \epsilon > 0 \), \( \exists M \) such that, if \( m > M \), \( |a_m - a| < \epsilon \).

Often described as a game - challenged with an \( \epsilon \), need to produce \( M \).
Simple examples:

\[ \lim_{i \to 0} \sum_{i=1}^{\infty} \frac{1}{i} = 1, 1/2, 1/3, \ldots \]  

Intuitively, this limit is 0. But how to prove this?

If challenged with \( \epsilon = 0.12 \), then

choose \( M = 9 \) since then \( a_M = \frac{1}{9} \) and \( |\frac{1}{9} - 0| = \frac{1}{9} = 0.111\ldots \)

For \( \epsilon = .12 \), any \( M > 9 \) would do (or even \( M = 8 \) if we use \( m > M \))

But we need to prove for all \( \epsilon \), so we need formula for choosing \( M \)

For any \( \epsilon \). In our case, \( \frac{1}{i} \) is strictly decreasing, so to get

\[ |\frac{1}{i} - 0| < \epsilon \quad \forall \quad i \geq M, \quad \text{we just need} \quad |\frac{1}{M} - 0| < \epsilon \]

i.e. pick any \( \frac{1}{M} > \frac{1}{\epsilon} \). \( \checkmark \)

Example 2: In \( \mathbb{R}^2 \),

\[ \lim_{i \to 0} \sum_{i=1}^{\infty} \left( \frac{i}{i+1} \right)^2 \]

Again, we're confident that limit is \((0)\)

Now,

\[ \left| \left( \frac{i}{i+1} \right) - (0) \right| \]

more complicated. It is equal to

\[ \sqrt{ \left( \frac{1}{i} - 0 \right)^2 + \left( \frac{i}{i+1} - 1 \right)^2 } \]

We could expand this, try to argue that \( i \geq M \), then

\[ \left| a_i - (0) \right| \leq \left| a_M - (0) \right| \]

then find formula for \( M \) in terms of \( \epsilon \) using expression.

Better: Prove a result that limit of points/vectors in \( \mathbb{R}^n \) converges if and only if it converges in each component. So reduce to question in \( \mathbb{R}^1 \).
Now in proof, we need to use that there is \( M_i \) s.t. \( m > M_i := M_i(\varepsilon_i) \).

\[
| (a_m)_i - a_i | < \varepsilon_i \quad \text{for each} \quad i = 1, \ldots, n .
\]

\( i \text{th component} \)

of \( a_m \in \mathbb{R}^n \)

To show that \( | \tilde{a}_m - a | \) can be made arbitrarily small for some \( M \).

\[
(\text{i.e. } \forall \varepsilon, \exists \tilde{a}_m \quad \text{s.t.} \quad | \tilde{a}_m - a | < \varepsilon)
\]

\[
| \tilde{a}_m - a | = \sqrt{((a_m)_1 - a_1)^2 + \cdots + ((a_m)_n - a_n)^2} \quad (\ast)
\]

If we choose \( M = \max_i M_i \), then we can guarantee each component \( \leq \varepsilon_i \)

\[
| \tilde{a}_m - a | \leq \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_n^2} .
\]

If we are given \( \varepsilon > 0 \), want to find \( M \) with \( (\ast) \leq \varepsilon \).

Then pick \( \varepsilon_i \)'s so that

\[
| (\varepsilon_i) | \leq \varepsilon
\]

One cute way to do this:

\[
\varepsilon_i = \frac{\varepsilon}{\sqrt{n}} , \quad \text{then} \quad | (\varepsilon_i) | = \varepsilon .
\]

(Style points for elegant choice...)

\[
\text{then desired}
\]

Alternately, choose \( \varepsilon_i = \varepsilon \) for \( i = 1, \ldots, n \), get

\[
| \tilde{a}_m - a | < \sqrt{n \cdot \varepsilon^2} = \sqrt{n} \cdot \varepsilon \quad \text{if} \quad m > M = \max_i M_i(\varepsilon)
\]

\[
\text{clear that since } \sqrt{n} \varepsilon \text{ can made arbitrarily small.}
\]

Proof of opposite direction in "iff" statement of Prop 1.5.13 is much easier. Leave you to read it.