On Monday, discussed definition of limit of function:

\[
\lim_{x \to x_0} f(x) = a \quad \text{if for every } \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ s.t. } \\
| x - x_0 | < \delta \Rightarrow | f(x) - a | < \varepsilon.
\]

In particular, if \( f: x \to \mathbb{R}^m \), \( x_0 \in \mathbb{R} \), say \( f \) is continuous at \( x_0 \) if \( \lim_{x \to x_0} f(x) = f(x_0) \). (and we say "continuous on \( X \" if continuous for all \( x_0 \in \mathbb{R} \).

Just as with limits of sequences, limits of functions converge if and only if each component function \( f_i(x) \), \( i = 1, \ldots, m \), converges.

Again, get big list of properties for limits of functions by combining this result with our knowledge of one-var. limits.

Thm. \( 1.5.23 + 1.5.24 \) limits of functions behave well under:

1. addition
2. "multiplication by scalar function" \( h: \mathbb{R}^n \to \mathbb{R} \) \( ( \lim h \cdot f ) \) vs. \( ( \lim h ) \cdot ( \lim f ) \)
3. dot product
4. composition \( ( \text{where both functions defined} ) \)

Thm. \( 1.5.28 + 1.5.29 \) same properties hold for continuous functions.

Cor: polynomials, rational functions, combinations of "elementary functions" are continuous.
Proposition: \( U: \text{open in } \mathbb{R}^n \), \( f: U \rightarrow \mathbb{R}^m \) is continuous if and only if the inverse image of every open set \( V \subseteq \mathbb{R}^m \) is open.

Interesting because didn't use the topology of open balls used to define open sets in \( \mathbb{R}^n \). Could define open sets differently and get different family of continuous functions, taking above as the definition. Try to prove it!

\( (\Leftarrow): \) Roughly, pick \( B \in f(x_0) \) it is open in \( \mathbb{R}^m \) so inverse image is open and contains \( x_0 \), so contains open ball around \( x_0 \), take this to be \( S \)-ball.

We could dive into derivatives now.

But we need one ingredient - Mean Value Theorem to prove a criterion for when a function is differentiable.

(among many other foundational results)

MVT: \( f: [a,b] \rightarrow \mathbb{R} \) continuous, differentiable on \( (a,b) \) then \( \exists \ c \in (a,b) \) such that \( f'(c) = \frac{f(b)-f(a)}{b-a} \).

\( \text{Recall } f'(c) = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} \), slope of tangent \text{ & \slope of secant line} \)

Common picture:

Plan: make function where desired \( C \) is maximum/minimum:

\[ h(x) := f(x) - \text{(secant line)} \]

Note \( h(a) = h(b) = 0 \) and continuous on \([a,b] \),

\[ f(a) + \left[ \frac{f(b)-f(a)}{b-a} \right] (x-a) \]
Now we need to show \( h \) has max or min. If \( h \equiv 0 \), then done.

If \( h \neq 0 \), then it achieves either max or min (or both) somewhere in \((a, b)\); call it \( c \). Since \( f \) differentiable on \((a, b)\), then \( h \) differentiable at \( c \). Since \( c \) is max/min, then \( h'(c) = 0 \).

(remember \( h'(c) = f'(c) - \left[ \frac{f(b) - f(a)}{b - a} \right] \))

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Two facts to prove:

1. if \( c \) is max/min of function \( h : (a, b) \rightarrow \mathbb{R} \) differentiable then \( h'(c) = 0 \).

2. \( h : [a, b] \rightarrow \mathbb{R} \) continuous

   then \( \exists c \in [a, b] \) with \( c = \text{max} \) i.e. \( h(c) \geq h(x) \) for all \( x \in [a, b] \)

   \( \exists c' \in [a, b] \) with \( c' = \text{min} \) i.e. \( h(c') \leq h(x) \) \( \forall x \in [a, b] \)

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(1) is easy. If \( c \) max, then \( \lim_{H \to 0} \frac{h(c+H) - h(c)}{H} = \begin{cases} \leq 0 \text{ if } H > 0 \\ \geq 0 \text{ if } H < 0 \end{cases} \)

But if \( h \) diff. at \( c \), then only possibility is that the limit = 0. /Same idea for min.

(2) is more subtle. Prove more generally for any \( f : C \rightarrow \mathbb{R} \) where

\( C \subseteq \mathbb{R}^n \) is a compact set.

This statement (2) will follow from Bolzano-Weierstrass fest.

We say a subset \( X \subseteq \mathbb{R}^n \) is bounded if it is contained in a ball of finite radius centered at \( \ldots \) the origin, say.

Then \( C \subseteq \mathbb{R}^n \) is said to be compact if it is closed and bounded.

(think of examples where one of these is satisfied, not the other)