On Wednesday, final ingredient before taking up derivatives is MVT.

proved it except for statement:

\[ f: [a,b] \to \mathbb{IR} \text{ continuous, then } f \text{ attains max/min on this closed interval.} \]

More general statement can be made, but need definitions.

Bounded: We say \( X \subset \mathbb{IR}^n \) is bounded if it is contained in a ball of finite radius. (H-H take ball centered at origin.) Why not?

Compact: We say \( C \subset \mathbb{IR}^n \) is compact if it is closed and bounded.

Note for example \((a,b)\) is bounded in \( \mathbb{IR} \), but not closed.

\([a,\infty)\) is closed in \( \mathbb{IR} \), but not bounded.

while \([a,b]\) is both i.e. compact.

Then more general statement (Theorem 1.6.9): \( C \subset \mathbb{IR}^n \) compact

\[ f: C \to \mathbb{IR} \text{ continuous \iff there } \exists M \in C \text{ s.t. } f(M) > f(x) \]

and \( \exists M \in C \text{ s.t. } f(M) < f(x) \text{ for all } x \in C. \]

To prove this, first prove Bolzano-Weierstrass Theorem.
Now we can state:

**Theorem (Bolzano-Weierstrass)** Every sequence on a compact set has a convergent subsequence \( \{ x_i \} \) whose limit lies in C.

**Proof:** C is assumed closed, so if \( \{ x_{i_k} \} \) converges, then must converge to a point in C. So we just have to show a convergent subsequence exists.

**Rough idea:** C is bounded, so finite volume. Infinitely many points in finite volume must accrue somewhere.

So put C inside big n-ball box in \( \mathbb{R}^n \). Form smaller boxes by cutting in half along all dimensions. In \( \mathbb{R}^2 \):

one of these boxes most contain infinitely many points.

We don't know which, but we just need to show a convergent subsequence exists.

Call this \( B_1 \). Pick \( x_{i_1} \) in \( B_1 \). Now chop all coordinates defining \( B_1 \) in half. Repeat argument, choosing \( B_2 \) with oo-many

many points and \( x_{i_2} \) in \( B_2 \).

We claim this sequence \( \{ x_{i_1}, x_{i_2}, \ldots \} \) converges.

Indeed, given any \( \epsilon > 0 \), we just find a box \( B_M \) that fits in a ball of radius \( \epsilon/2 \).

so that \( |x_m - y| < \epsilon \), if \( m > M \).

Boole proof is even more concrete by choosing boxes that shrink by \( 10^{th} \)'s, so you can find of each box as giving decimal expansion of eventual limit.

\[ \text{Note: technical issue about whether it include body of each square} \]

Suppose this was \( B_i \).

What is \( a \) by the way? We don't know. We just know it lives in all of the boxes.
Book has nice example illustrating issues with Bolzano-Weierstrass

Some sequences are easy to analyze. E.g. $\{ \frac{1}{n} \}_{n=1}^{\infty}$ or even $\{1, 1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \cdots \}$ but $\sin(x_n)$ for any sequence of real #s $x_n$ is an $\infty$ sequence in $[-1,1]$ so has conv. subsequence by Bolzano-Weierstrass Thm.

However, if we try to find which cuts contain only $\infty$ many pts, find it is very hard to do.

E.g. $x_n = 2 \cdot 10^n$

Then $\sin(x_n)$ pos/neg. depending on $n$th digit of $\pi$. Hard question...

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On to proof of Thm. 1.6.9.

Recall that a number $S$ is the supremum of $f: \mathbb{R} \to \mathbb{R}$ if it is the least upper bound of the values $\{f(x) | x \in U\}$

Write $S = \sup_{x \in U} f$.\[ upper \ bound: \ a \ number \ a \ s.t. \ a > f(x) \ \forall \ x \in U. \]

least upper bound: if $b$ is any other upper bound, $a \leq b$.

There is a corresponding notion of greatest lower bound.

Property of real numbers: Every nonempty subset $X \subseteq \mathbb{R}$ with upper bound has a least upper bound. (Thm 0.5.3) in book.

Another statement for lower bounds.