Integration theory. In one-variable integration, definition of (definite) integral was via Riemann sum. In pictures, calculating area under curve by successively finer approximations:

\[ a \quad b \]

Also have a choice of where to sample, sometimes test point noted \( x_i^{*} \in [x_{i-1}, x_i] \)

(left endpoints / right endpoints / midpoints / minima / maxima)

Lower sum \quad Upper sum

Expectation: if \( f \) is nice enough, and partition width \( \to 0 \) as \( n \to \infty \), all these sums converge to same real number, this is value of integral.

Miracle of FTC: if \( f \) is really nice (elementary function), we can find this limit exactly using anti-derivative \( F \) of \( f \). "indefinite integral"

One peculiarity of Riemann sums: signed area. So \( \int_{a}^{b} f = -\int_{b}^{a} f \).

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^{*})(x_i - x_{i-1})
\]
signed distance. Could have put \( |x_i - x_{i-1}| \) instead.
That issue becomes much more complicated in higher dimensions, so
("positive orientation") initially we're aiming for notion of absolute
area/volume. Still could be negative
depending on \( f < 0 \)
or \( f > 0 \).

Change in point of view:
\[
\int_a^b f(x) \, dx = \int_{\mathbb{R}} g(x) \, dx
\]

or \( |dx| \)

Denote this by \( |dx| \)
to distinguish from \( dx \).

Slightly more verbose:
Book uses \( |dx^n| \)
for integrals over \( \mathbb{R}^n \).

\[
\begin{array}{c}
\begin{tikzpicture}
\draw[->] (-1,0) -- (1,0);
\draw (0,0) -- (0,1);
\draw[thick] (-1,1) .. controls (-0.5,0) .. (0.5,1);
\draw[thick] (0.5,1) .. controls (1,0) .. (1.5,1);
\draw[thick] (1.5,1) .. controls (1,0) .. (0.5,1);
\draw[thick] (0.5,1) .. controls (0,0) .. (-0.5,1);
\draw[thick] (-0.5,1) .. controls (-1,0) .. (-1.5,1);
\draw[thick] (-1.5,1) .. controls (-1,0) .. (-0.5,1);
\draw (0,1) node[above] {f};
\end{tikzpicture}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\draw[->] (-1,0) -- (1,0);
\draw (0,0) -- (0,1);
\draw[thick] (-1,1) .. controls (-0.5,0) .. (0.5,1);
\draw[thick] (0.5,1) .. controls (1,0) .. (1.5,1);
\draw[thick] (1.5,1) .. controls (1,0) .. (0.5,1);
\draw[thick] (0.5,1) .. controls (0,0) .. (-0.5,1);
\draw[thick] (-0.5,1) .. controls (-1,0) .. (-1.5,1);
\draw[thick] (-1.5,1) .. controls (-1,0) .. (-0.5,1);
\draw (0,1) node[above] {g};
\end{tikzpicture}
\end{array}
\]

Mored complexity of definition of domain into definition
of \( g \). In particular, \( g \) not continuous on \( \mathbb{R} \). Lead to prettier
definition.

With this in mind, make some initial assumptions about which function
we try to integrate:
\[ f : \mathbb{R}^n \to \mathbb{R} \]

1. Want \( |f| \) to be bounded.
2. Want \( \text{Supp}(f) = \{ x \in \mathbb{R}^n \mid f(x) \neq 0 \} \) bounded.

Remember, bounded sets are those contained in \( B_r(0) \) for some \( r \).
then, if integral is defined, then \( \int_{\mathbb{R}^n} |f(x)| \, dx < \infty \).

Finally, how to define Riemann integration for \( \mathbb{R}^n \)? Partition \( \mathbb{R}^n \) into pieces. Always do same partition \( \to \) cubes with side 1, cubes with side \( \frac{1}{2} \), ..., cubes with side \( \frac{1}{2^n} \).

Really particular - label them consistently using \( n \)-tuples of integers of width \( \frac{1}{2^n} \). So need \( 2^n \) cubes to get to \((1,0,\ldots,0)\)

starting at \((0,\ldots,0)\).

Definition: \( f \) is integrable if upper sum \( f \) = lower sum \( f \) using limit of dyadic pavings. (and then say \( \int_{\mathbb{R}^n} f \, dx = \) upper sum \( f \))

Example: \( \int_0^1 x \, dx \)

Think of this as \( \int g(x) \, dx \) where \( g(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 0 & \text{else} \end{cases} \)

For each \( N \), chop up \([0,2] \) into intervals ("1-cubes") of width \( \frac{1}{2^n} \)

\[ N = 1: \]

\[ \text{lower sum: } \sum_{\text{intervals}} g(\text{min in interval}) \cdot (\text{volume of interval}) \]

In dyadic paving, volume is always constant \( = \left( \frac{1}{2^n} \right)^n \) \( n \): dim.
lower sum:  \[ \sum_{k=0}^{3 \cdot 2^N-1} g(\frac{k}{2^N}) \cdot \frac{1}{2^N} = \frac{1}{2^N} \sum_{k=0}^{3 \cdot 2^N-1} \frac{k}{2^N} \]

Similarly, upper sum:
\[ \sum_{k=0}^{3 \cdot 2^N-1} g(\text{max on interval } k) \cdot \frac{1}{2^N} = \frac{1}{2^N} \sum_{k=0}^{3 \cdot 2^N-1} \frac{k+1}{2^N} \]

We have
\[ \sum_{k=0}^{3 \cdot 2^N-1} k = \frac{(3 \cdot 2^N-1)(3 \cdot 2^N)}{2} \]

(multiplied by \( \frac{1}{2^N \cdot 2^N} \))
in lower sum

so lower sum:
\[ \lim_{N \to \infty} \frac{9}{2} \frac{(2^N-\frac{1}{3}) \cdot 2^N}{2^N \cdot 2^N} = \frac{9}{2} \]

\[ \to 1 \text{ as } N \to \infty \]

Similarly upper sum:
\[ \frac{3 \cdot 2^N (3 \cdot 2^N+1)}{2} \frac{1}{2^N \cdot 2^N} \to \frac{9}{2} \text{ as } N \to \infty \]

Immediate remark: If we can find criterion on \( g \) such that \( g \) is integrable

(i.e. \( \int g \, d^N x = \text{upper sum} \) and/or

\( \int g \, d^N x = \text{lower sum} \)

then you can compute integral using dyadic
tiling and any sample points. This is

just because, if \( x^k \) are sample points in interval \( k \)

( or cube)

then for all \( k \)
\[ g(\text{min on } k^{th} \text{ interval cube}) \leq g(x^k) \leq g(\text{max on } k^{th} \text{ cube}) \]

so \[ \sum_{k \text{ in } C_N} g(x^k) \cdot \left( \frac{1}{2^N} \right)^n \leq U_N(g) \]

taking limits and using squeeze theorem shows this sampling for Riemann sum gives same \( f \).