Definition of paving: A subset $X \subseteq \mathbb{R}^n$ is paved by $P = \{P_i\}$ if $P_i$ are bounded subsets of $X$ such that:

1. $\bigcup_i P_i = X$
2. $\text{vol}_n(P_i \cap P_j) = 0$ if $P_i \neq P_j$ (only)
3. Any bounded subset of $X$ intersects finitely many $P_i$.
4. $\forall P_i \in P$, $\text{vol}_n(\partial P_i) = 0$.

Then we want notion of shrinking set of nested pavings. "Nested partition" $P_N$ is a sequence of pavings of $X \subseteq \mathbb{R}^n$ if every subset in $P_{N+1}$ is contained in a subset of $P_N$.

The subsets of $P_N$ shrink as $N \to \infty$: Precisely, $\lim_{N \to \infty} \sup_{P_i \in P_N} \text{diam}(P_i) = 0$

(\text{diam}(P_i) = \sup_{x,y \in P_i} |x-y| is the "diameter of $P_i$".)

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(Independence of pavings by nested partitions)

If $f: \mathbb{R}^n \to \mathbb{R}$ is $\text{L}_N$-integrable (w.r.t. dyadic paving) then

$$\lim_{N \to \infty} U_{P_N} (f \cdot 1_X) = \lim_{N \to \infty} L_{P_N} (f \cdot 1_X) = \int_X f(x) \, d^n x$$

Conversely, if upper and lower sums for $P_N$ agree, then $f$ integrable.
pf of independence of pavings:

First check $1_X$ is payable by $\xi \mathcal{F}_N$. True because of weird way we defined nested partition of $X$. So that $\partial X$ is contained in $\partial \mathcal{P}_i$ with $\mathcal{P}_i \in \mathcal{F}_N$ for any $N$.

By definition, this is a set of measure 0. (So having nested partition of $X$ means $\partial X$ is somewhat nice)

Now main issue: how to compare dyadic pavings with arbitrary pavings? Want to relate their upper and lower sums.

In $\mathbb{R}^2$:

\[
\text{Issue: If paving $\mathcal{F}_N$ shares many dyadic cubes, hard to compare to Riemann sum over $D_N$.}
\]

\[
\text{Idea: choose } N'' \gg N \text{ so that "most" of } \mathcal{P}_i \in \mathcal{F}_N \text{ are entirely inside cubes } D_N. \]

Write $U_{\mathcal{F}_N''}(f)$

\[
= \sum_{\mathcal{P}_i \cap \partial D_N = \emptyset} \sup_{\mathcal{P}_i}(f) \cdot \text{vol}_n \mathcal{P}_i + \sum_{\mathcal{P}_i \cap \partial D_N \neq \emptyset} \sup_{\mathcal{P}_i}(f) \cdot \text{vol}_n \mathcal{P}_i
\]

\[
\text{This is bigger than } - \sup_{\mathcal{P}_N''} |f|
\]

\[
\text{need to make } \sum_{\mathcal{P}_i \cap \partial D_N \neq \emptyset} \text{vol}_n \mathcal{P}_i \text{ small. (choose } N'' \text{ so that } D_N \text{ small)}
\]
Determinants: Give a recursive definition or axiomatic definition.

(de determinants give volume expansion by action of linear map)

Recursive: Illustrate first column expansion recursive definition.

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \text{ then } \det(A) = 1 \cdot \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} \leftarrow \text{ take det. of submatrix of rows, columns away from } a_{1,1} \]

\[ + 4 \cdot \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} \leftarrow \text{ signs alternate} \]

So propose \( 1 \cdot \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} \) as \[ \begin{pmatrix} 1 & 2 & 4 \\ 5 & 6 & \end{pmatrix} \]

You can take as definition the expansion along any fixed column. Not yet clear that they all produce same #. (be careful with signs)

Book uses symbol \( \Delta_n \) for det. of \( n \times n \) matrix.

\[ \Delta_1([a]) = a. \quad \Delta_n(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \cdot \Delta_{n-1}(A_{\cdot\cdot\cdot}) \]

Thus: \( \Delta_n \) is the unique function \( f((\mathbb{R}^n)^n) \rightarrow \mathbb{R} \)

\[ \text{n-tuples of vectors in } \mathbb{R}^n \]

s.t.

1. \( f \) is linear in all components (i.e. \( n \times n \) matrix)
2. \( f \) is antisymmetric
3. \( f(I_n) = 1. \) (Better: \( f(e_1, \ldots, e_n) = 1. \)