On Tuesday, learned that uniform convergence $f_k \to f$ is great, but
best: Dominated convergence theorem — Pick $R$ with
\[
|f_k(x)| \leq R \quad \forall x, \forall k,
\]
then if \( \lim_{k \to \infty} f_k = f \)
integrable
\[
\text{Supp } f_k(x) \subseteq B_R(0)
\]
then \( \lim_{k \to \infty} \int_{\mathbb{R}^n} f_k(x) \, d^n x = \int_{\mathbb{R}^n} f \)

Proposed definition of "Lebesgue integral" as applied to infinite sum of
integrable $f_k$:

If $f \triangleq \sum_{k=1}^{\infty} f_k$ with $\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(x)| \, d^n x < \infty \quad \text{(*)}$

then $\sum_{k=1}^{\infty} f_k$ converges to $f$ almost everywhere. Define

\[
\int_{\mathbb{R}^n} f(x) \, d^n x \overset{\text{def}}{=} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(x) \, d^n x
\]

To make sure well-defined, if $f_k, g_k$ satisfy $(\ast)$ and $\sum_{k} f_k = \sum_{k} g_k$
so converge to same function up to measure 0.

then their Lebesgue integrals are equal.

To prove it, we need dominated convergence theorem.
0-th example: \( f_1 = f \) : Riemann integrable, \( f_2 = f_3 = \ldots = 0 \).

then Lebesgue of \( \sum_{k=1}^{\infty} f_k = \) Riemann int. of \( f \).

1st example: \( \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+1}} \, d^n x \). If \( n = 1 \), \( \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \pi \).

Not Riemann integrable because, though output is always \( \leq 1 \), support is not bounded.

Need to use infinite series \( \sum_{k=1}^{\infty} f_k \), with each \( f_k \) of bounded support.

E.g., since function is constant on sphere \( |x| = r \), break \( \mathbb{R}^n \) into space between spheres of radius \( 2^k \).

\( f_1 := \frac{1}{1 + |x|^{n+1}} \) : 1 unit ball

\( f_2 := \frac{1}{1 + |x|^{n+1}} \) : 1 ball of radius 2

\( f_3 := \frac{1}{1 + |x|^{n+1}} \) : ball of rad. 1.

To show this is Lebesgue integrable, need to show:

\[ \sum_{k=1}^{\alpha} \int f_k(x) \, d^n x < \infty. \]
Need to show: $\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(x) |d^n x| = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} g_k(x) |d^n x|$

Prove difference is 0. So show

$$\lim_{l \to \infty} \sum_{k=1}^{l} \int_{\mathbb{R}^n} (f_k - g_k)(x) |d^n x| = 0$$

can be made arbitrarily small. (i.e. smaller than any $\epsilon > 0$)

Rewrite:

$$\lim_{l \to \infty} \int_{\mathbb{R}^n} H_l(x) |d^n x|$$

Done if we can bring limit inside. If $H_l$ satisfies conditions of Dominated Convergence Theorem (i.e. we can find suitable $R$) Picture in $\mathbb{R}^1$: supported on $[-R, R]$ with values in $[-R, R]$ then done.

$H_l$'s won't necessarily satisfy this, but truncated version does.

Order: Make tail $\sum_{k=M}^{\infty} \int |h_k|$

small, say $\delta < \epsilon$ for

Sufficiency large $M$. Then pick $R$ depending on $H_M$

then for any $l > M$:

$H_l = H_l - H_l \text{trunc}(R) + \text{trunc}(R)$

$H_l - H_M - H_l \text{trunc}(R) + H_M = 0$

by dom-conv.

$H_l - H_M \leq \epsilon$ controllable

$\text{as } h_p \text{ are Riemann integrable}$
Clever idea: Map $B_k$ : volume between sphere of radius $2^{k-1}$ and that of $2^{k-2}$.

$f_1$ is different, since unit ball topologically different from annuli.

Map all annuli to annulus between $R=1$ and $R=2$. $\tilde{f}_k : x \mapsto 2^{k-2} x$ maps this annulus to support of $f_k$.

\[ \int_{B_k} |f_k(x)| \, d^n x = \int_{B_k} \frac{1}{1 + |x|^m} \, d^n x \]

\[ = \int_{B_2} \left( \frac{1}{1 + |x|^m \cdot \tilde{f}_k(x)} \right) \left| \text{det} D\tilde{f}_k(x) \right| \, d^n x \]

\[ = \int_{1 + 2^{k-2} \cdot 1^m} \left( \frac{1}{1 + |x|^m \cdot \tilde{f}_k(x)} \right) \left| \text{det} D\tilde{f}_k(x) \right| \, d^n x \]

\[ \leq \int_{B_2} 2^{\frac{(k-2)n}{(k-2)m}} \frac{1}{1 + |x|^m} \, d^n x \]

Need this sum over $k$ to converge. Geometric series so need $m > n$. 

\[ \int_{B_k} \]