Differentiating under integral sign:

Define \( F(t) = \int_{\mathbb{R}^n} f(t, x) \, dx \) \( f(t, x) : \mathbb{R}^{n+1} \to \mathbb{R} \) integrable for any fixed \( t \).

If \( \frac{d}{dt} f(t, x) \) exists a.e. \( x \) and (for all \( t_1, t_2 \))

\[ \exists s > 0 \text{, integrable } g \text{ s.t. if } |t_1 - t_2| < s \text{ then } \left| \frac{f(t_1, x) - f(t_2, x)}{t_1 - t_2} \right| \leq g(x) \],

then \( F \) is differentiable with derivative \( \frac{d}{dt} F(t) = \int_{\mathbb{R}^n} f(t, x) \, dx \).

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Do simple example.

If compact region, then (*) easily satisfied if \( \frac{d}{dt} f(t, x) \) is continuous.

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**Fourier transform:** \( \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{ix\xi} \, dx \)

\( i = \sqrt{-1} \).

\( e^{ix\xi} = \sum_{k=0}^{\infty} \frac{(i\xi)^k}{k!} = \cos \xi x + i \sin \xi x \). If you prefer just think of writing function \( g = g_1 + ig_2 \) with \( g_1, g_2 \) real.

Analyze behavior of Fourier transform under differentiation.
If we can differentiate under integral sign, then
\[
\frac{d}{d\xi} \hat{f}(\xi) = \int_{\mathbb{R}} \frac{d}{d\xi} \left( f(x) e^{ix\xi} \right) dx = -i\xi f(x) e^{ix\xi}
\]
(see this from power series or Euler's identity)

check this is possible exactly when \( x \mapsto |x f(x)| \) is integrable. (4)

Moral: \( \hat{f} \) differentiable when \( |x f(x)| \) integrable (statement about decay of \( f \) at \( \infty \). Can't go to \( \infty \) like \( 1/x \) for example.

So Fourier transform: behavior at \( \infty \) of \( f \) \( \leftrightarrow \) smoothness (i.e. diff of \( \hat{f} \)).

Also analyze \( \hat{f}' \) by parts, get
\[
\hat{f}'(\xi) = -i\xi \hat{f}(\xi)
\]
so Fourier transform turns differentiation into multiplication.

(4) check:
\[
e^{i(\xi+h)x} - e^{ix\xi} = \frac{e^{i\xi x} e^{ihx} - 1}{h}
\]
need to bound this for small \( h \).

\[
e^{ihx} = \cos hx + is\sin \frac{hx}{h} \to 0 \quad \text{as} \quad h \to 0
\]
with size \( h \).

We're diving back into manifolds. What are manifolds?

- our notion of smooth manifold = locally graph of \( C^1 \) function
- locally zero locus of some \( F : \mathbb{R}^n \to \mathbb{R}^{n-k} \)
  - \( k \)-manifold: \( f : \mathbb{R}^k \to \mathbb{R}^{n-k} \)
  - \( k \) free vars, \( n-k \) dependent vars.

(key condition of implicit function theorem)

Less time on: parametrizations of manifolds.

\( \gamma : U \subset \mathbb{R}^k \to M \) (e.g. parametric curves attempt to parametrize one-manifolds)

s.t. 1. \( U \) open
2. \( \gamma \) is \( C^1 \), one-one, onto \( M \)
3. \( [D\gamma(u)] \) is one-one \( \forall u \in U \).

(condition 1 prevents \( \gamma : (0, 2\pi) \to \mathbb{R}^2 \) from being valid param of unit circle.)

Very hard to give parametrizations.

(two sides of non-linear transformation — non-linear kernel is zero locus
- non-linear image is parametrization)

But this is exactly what we need in order to define integral on manifold.

(See later that issues of circle parametrization ok since failure is on a set of measure 0)
What is volume of as subset of 2-manifold?

First small step in this direction - volumes of parallelograms.

Example: In $\mathbb{R}^3$, two vectors $\mathbf{u}, \mathbf{v}$ in the plane containing $\mathbf{u}, \mathbf{v}$. What is its volume? (as 2-dim manifold)

In Ch.4, we learned volume of $k$-parallelogram in $\mathbb{R}^k$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is $|\det(\mathbf{v}_1, \ldots, \mathbf{v}_k)|$

Clearly that doesn't work here: $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

gives a $2 \times 3$ matrix.

Clever fix: \( \text{Vol}_k P(\mathbf{v}_1, \ldots, \mathbf{v}_k) = |\det(\mathbf{v}_1, \ldots, \mathbf{v}_k)| \)

\[ = \sqrt{\det(T^T T)} \]

\[ T = \begin{bmatrix} v_1 & \cdots & v_k \\ 1 & \cdots & 1 \end{bmatrix} \]

(remember $\det(T^T T) = \det(T^T) \det(T)$)

\[ = \det(T)^2 \cdot | \]

this definition makes sense regardless of whether $T$ square itself so $T^T$ is $n \times m$ matrix.
What does multiplication look like? If we have $k$ vectors in $\mathbb{R}^n$:

\[
\begin{bmatrix}
  v_1 & \cdots & v_k \\
  v_1^T & \cdots & v_k^T
\end{bmatrix}
\]

in terms of $|v_i|$, $|v_j|$ and cosines of angles between them in $\mathbb{k}$-plane.

(So independent of location of vectors in space — “anchor” as the book puts it.)

Book calls it definition. Not a definition.

Show it agrees with volume in $\mathbb{k}$-space if we take $v_1, \ldots, v_k$ as basis for $\mathbb{R}^k \subset \mathbb{R}^n$.

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Plan: Given parametrization $\gamma : U \subset \mathbb{R}^k \rightarrow M \subset \mathbb{R}^n$ suitable of $k$-manifold $M$ in $\mathbb{R}^n$ pave $U$ with cubes weird shapes.

But linearizing $D\gamma : \text{cubes} \rightarrow \text{parallelograms}$ use these to approximate volume in $M$.

First, need to relax our notion of parametrization. Pre-conditions:

1. boundary of $U$ is of $k$-dim vol $0$
2. $\gamma(U) = M$
3. $\exists$ $X \subset U$ with $k$-dim vol $0$ s.t. $\gamma(U - X) \subset M$
4. $D\gamma(y)$ is one-one $\forall y$ in $U - X$. $\gamma(X) \cap C$ has $k$-vol $= 0$ $\forall$ compact $C \subset M$. $\gamma : U - X \rightarrow M$ is one-one, $C^1$ function with locally Lipschitz derivative.