On Wednesday, defined integration using Riemann sums on dyadic pavings: If $g$ is integrable then $g : \mathbb{R}^n \to \mathbb{R}$

$$\int_{\mathbb{R}^n} g \, |d^nx| = \lim_{N \to \infty} \sum_{\text{dyadic paving of level } N} g(x_i^*) \, \text{vol (i}^{\text{th}} \text{cube)}$$

(with the same constant for all cubes: $(\frac{1}{2^N})^n$)

Recall these are half-closed cubes so disjoint. (draw $n=2$ again)

Easiest functions to integrate: constants.

$$\int_a^b 1 \, dx = \left[ x \right]_a^b = b-a$$

 FTC

area $\frac{1}{2} (b-a)$

roughly: $(b-a)2^n - 1$ cubes.

True if $a-b$ is integer, for example.

$$\lim_{N \to \infty} \frac{1}{2^N} \sum_{i=0}^{2^N} g(x_i^*) \to b-a.$$ 

Riemann sum

Now write it as:

$$\int_{\mathbb{R}} 1_{[a,b]} \, |d^nx|$$

where $1_{[a,b]} = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{else} \end{cases}$

"Characteristic function of the set $[a,b]$"

"Indicator function" in book

In higher dimensions, given a set $A$

$$\int_{\mathbb{R}^n} 1_A(x) \, |d^nx|$$

is very interesting!

already much cooler for $n=2$. 
In fact, some sets $A$ have $1_A$ not integrable, even in $\mathbb{R}^1$. e.g. $A = \text{irrationals in } [0,1]$. Book says a set $A$ is "pavable" if $1_A$ is integrable.

Basic facts about $\text{vol}(A) := \int_{\mathbb{R}^n} 1_A(x) \, d^n x$:

**Theorem:** If $A, B$ disjoint, pavable, then $A \cup B$ is pavable and $\text{vol}(A \cup B) = \text{vol}(A) + \text{vol}(B)$.

**Proof:** First write out definitions!  $\text{vol}(A \cup B) = \int_{\mathbb{R}^n} 1_{A \cup B} \, d^n x$  $\text{vol}(A) = \int_{\mathbb{R}^n} 1_A \, d^n x$;  $\text{vol}(B) = \int_{\mathbb{R}^n} 1_B \, d^n x$.

In section, you proved with Theo that  $\int_{\mathbb{R}^n} f + g = \int_{\mathbb{R}^n} f + \int_{\mathbb{R}^n} g$ (squeeze theorem with upper/lower Rieman sums.)

so result follows upon noting that $1_{A \cup B} = 1_A + 1_B$ if $A, B$ disjoint.

**Application:** Volume is translation invariant. Given set $A$, vector $v$ pavable.

then $\text{vol}(A + v) = \text{vol}(A)$.

In particular $A + v$ pavable.

**proof:** To show $A + v$ integrable, need to compute upper, lower Riemann sums.

What is $\text{L}(\frac{1_{A+v}}{2N})$?  $\lim_{N \to \infty} (\frac{1}{2N})^n \sum_{\text{cubes in } A+v} \frac{1}{\text{cubes in } A+v}$

we mean: cubes entirely contained in $A + v$.

Since $L_N$ is increasing in $N \to \infty$ we have...
\[ L(1_{A+V}) \geq \sum_{\text{level } N \text{ cubes } C \text{ in } A+V} \frac{1}{C+V} \]

C is again a cube. This is integrable with volume \( \text{vol}(C) = \left( \frac{1}{2^N} \right)^n \)

So \[ L\left( \sum 1_{C+V} \right) = \sum \int 1_{C+V} = \sum \int C+V \]

\[ = \sum \text{vol}(C) \]

So \[ L(1_{A+V}) \geq \sum_{\text{level } N \text{ cubes } C \text{ in } A} \text{vol}(C) \text{ for each N} \]

\[ \Rightarrow L(1_{A+V}) \geq \lim_{N \to \infty} \sum_{\text{level } N \text{ cubes } C \text{ in } A} \text{vol}(C) = L(1_A). \]

Similarly, show upper bound for \( U(1_{A+V}) \) using cubes with \( \neq \emptyset \) intersection with \( A \).

Then get \[ L(1_A) \leq L(1_{A+V}) \leq U(1_{A+V}) \leq U(1_A). \]

But outer two quantities are equal since \( \emptyset \) assumed parallele.
volume are important application but we can, of course, consider more interesting integrable functions other than characteristic functions.

Two viewpoints: Given \( f \),

1. \[
\int_{\mathbb{R}} f \, d^4x 
= 
\int_{\text{supp}(f)} f 
\]

or

\[
\int_{\mathbb{R}^2} f \, d^2x 
\]

"area below \( f \)

2. As weighting factor

\[
\int_{\mathbb{R}^2} \text{assign value to each chunk} 
\]

"volume beneath surface \( f \)"

"density function" where \( \text{supp}(f) \) is our material

and \( f \) is density at any point.

Physical applications: Mass (\( A \)) = \[
\int_{A} \mu(x) \, d^n x 
\]

\( A = \mathbb{R}^n \).

\( \mu \): density function

Center of gravity (\( A \)) = \[
\int_{A} x_i \mu(x) \, d^n x 
\]

( or center of mass )

in direction \( x_i \) in \( \mathbb{R}^n \).