On Friday, reminded you about two ways of defining smooth manifold.

1) locally as zero locus of $F : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$

2) globally as parametrization $\gamma : U \subset \mathbb{R}^k \rightarrow \mathbb{M} \subset \mathbb{R}^n$.

Example: helix $\gamma : t \in \mathbb{R} \rightarrow \left[ \begin{array}{c} \cos t \\ \sin t \\ t \end{array} \right] \in \mathbb{R}^3$

(parametric curve)

defines smooth 1-manifold.$( \text{Jacobian matrix not } \frac{\partial}{\partial t} \quad \forall t. )$

so $D\gamma$ is one-one, for all $t$.

Plan: use parametrizations to compute volumes of manifolds (in our example, 1-dim' volume should be arc length of part of helix inside $\mathbb{R}^3$.)

Define $\text{vol}_k(M) = \int \left| d^k x \right|$.

$\gamma(U)$

$= \int_U \text{vol}_k(P_{\gamma(u)}(D_1\gamma(u), \ldots, D_k\gamma(u))) \left| d^k u \right|$

Better: Riemann sums like

$$\lim_{N \to \infty} \sum_{C \in D_N(\mathbb{R}^k)} \text{vol}_k(\gamma(cn_ku)) \approx \lim_{N \to \infty} \sum_{C \in D_N(\mathbb{R}^k)} \text{vol}_k(P_{\gamma(u)}(\ldots))$$
Issue #1: What is \( k \)-volume of parallelogram spanned by \( k \) vectors \( v_1, \ldots, v_k \) in \( \mathbb{R}^n \)?

Not \( |\det(v_1, \ldots, v_k)| \), since doesn't make sense.

Trickier: \( \sqrt{\det(T^T T)} \) where \( T = \left( \begin{array}{c} v_1 \cdots v_k \end{array} \right) \)

\( T \) matrix is square.
\( T \) is \( n \times k \).
\( T^T \) is \( k \times n \).
So \( T^T T \) is \( k \times k \).

Example: \( T = \left( \begin{array}{cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array} \right) \) (two vectors in \( \mathbb{R}^3 \) spanning plane)

then \( \det(T^T T) = \det\left( \begin{array}{cc} -v_1 & -v_2 \\ -v_2 & -v_1 \end{array} \right) \left( \begin{array}{c} 1 \cdots 1 \end{array} \right) = \det\left( \begin{array}{cc} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{array} \right) \)

all dot products:

\( v_i \cdot v_j = |v_i| |v_j| \cos \theta \quad \theta \): angle between \( v_i, v_j \)

in plane spanned by them.

independent of "anchor" coordinates

- location of parallelogram in space.
also independent of embedding of \( \mathbb{R}^b \) in \( \mathbb{R}^n \).

Back to our example: \( \gamma: t \mapsto \left[ \begin{array}{c} \cos t \\ \sin t \\ t \end{array} \right] \) then \( D\gamma = \left[ \begin{array}{c} \frac{d}{dt} (\cos t) \\ \frac{d}{dt} (\sin t) \\ 1 \end{array} \right] \)

\( D\gamma(t)^T D\gamma(t) = \gamma'(t) \cdot \gamma'(t) \)

\( = \sin^2 t + \cos^2 t + 1 = 2 \).
so arc length of one rotation of helix

\[ \int_0^{2\pi} \sqrt{2} \, dt = 2\pi \sqrt{2}. \]

(or up to any height \( T = 2\pi T \).)

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Issue #2: Very hard to find perfect parametrization \( ( U \) open, with \( \gamma : U \to M \) one-one, onto, ... )

So we relax definition as in p.3 of Friday's notes:

1. \( \partial U \) has \( k \)-diml volume 0
2. \( \gamma(U) \supseteq M \)
3. \( \exists x \in U \) with \( k \)-diml vol. 0 st. \( \gamma(U-x) \subset M \).
4. \( \gamma : U-x \to M \) is one-one, \( C^1 \) function with locally Lipschitz derivative.
5. \( D\gamma(x) \) is one-one \( \forall x \in U-x \).
6. \( \gamma(x) \cap C \) has \( k \)-volume 0 \( \forall \text{ compact } C \cap M \).

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Basic idea: pushed all "trouble spots" in failing parametrizations into a set of vol. 0, \( x \), in examples, \( x \) usually includes \( \partial U \) (check that circle can be parametrized using relaxed parametrization).
Example: parametrizing a cone. Cone in $\mathbb{R}^3$ given by $x^2 + y^2 - z^2 = 0$.

(technically not a manifold. Not graph at origin.)

associate manifold $M = \left\{ \left( \frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}, z \right) \mid x^2 + y^2 - z^2 = 0, \ z \in (0,1)^2 \right\}$

How to parametrize it?

$$\gamma: \left( r, \theta \right) \rightarrow \left( r \cos \theta \ , \ r \sin \theta \ , \ r \right)$$

$$\left[ 0,1 \right] \times \left[ 0,2\pi \right] \rightarrow \mathbb{R}^3$$

$$\gamma$$

Set $X = \partial M$ so that $U - X = (0,1) \times (0,2\pi)$

boundary of $U^\ast$

claim: their volume is 0.

which ensures that if $0 \leq m \leq k \leq n$ then

$\kappa$-volume of $m$-manifold is 0.

$H^n \times \mathbb{R}^2$ example:

$$T = \begin{pmatrix} V_1 & V_2 \\ 1 & 1 \end{pmatrix}$$

with $V_1 = (v_{1,1}, v_{2,1}, \ldots, v_{n,1})$,

$V_2 = (v_{1,2}, v_{2,2}, \ldots, v_{n,2})$

then regardless of $n$, $\det (T^T T) = \det \begin{pmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{pmatrix}$

$$= |v_1|^2 |v_2|^2 - (v_1 \cdot v_2)^2 = \frac{|v_1|^2 |v_2|^2 (1 - \cos^2 \theta)}{\sin^2 \theta}$$

Square root gives familiar form of $n$-gram vol.