We define k-manifold volume for relaxed parameterization

\[ \gamma: U \times X \to M \leq \mathbb{R}^n \] by

\[ \text{Vol}_k(M) = \int_{U \times X} \sqrt{\det(D\gamma(y)\top D\gamma(y))} \ |d^k u| \]

can also insert function here \( f(\gamma(y)) \)

and if \( f \cdot \sqrt{\det(\cdots)} \) is integrable,

then can compute weighted volumes.

Theoretical consideration:

Given two parametrizations, show they give same volume integral

(Answer shouldn't depend on how we enumerate points on manifold.

ease of curve - parametrization determines how fast you traverse curve.)

Proof: given two rel parametrizations

\[ \gamma_1: U \to M \quad \gamma_2: V \to M \quad U, V \leq \mathbb{R}^k \]

\[ \gamma_1 = \gamma_2 \circ \gamma_2^{-1} \circ \gamma_1 \]

\[ \Psi: U \to V = \text{"change of var"} \]

By chain rule. (*)

\[ \int_V \sqrt{\det(D\gamma_2(\Psi(y))\top D\gamma_2(\Psi(y)))} \ f(\gamma_2(\Psi(y))) \ |d^k y| \]

\[ = \int_U \sqrt{\det(D\gamma_2 \circ \Psi(y)\top D\gamma_2 \circ \Psi(y))} \ f(\gamma_2(\Psi(y))) \ |d^k u| \]
\[
\text{Expand } \det(D\xi(x)) = \sqrt{\det D\xi^T D\xi} \quad \text{do some rearranging, apply chain rule. done.}
\]

May have noticed we worked this as pair of perfect parameterizations, not relaxed ones.

\[\Phi: U \rightarrow V \text{ must be bijective, class C}', \text{ with } \Phi, \Phi^{-1} \text{ having Lipschitz derivatives.}\]

Book has nice example of issues at bad points in parametrization: two parameterizations of sphere using different poles \((P_1, P_1')\) vs. \((P_2, P_2')\)

\[\Phi = \gamma_2^{-1} \circ \gamma_1\]

if \(P_2, P_2'\) are poles of \(\gamma_2\) then a single point maps to \(P_2\) under \(\gamma_1\).

so consider \(\gamma_1^{-1}(P_2)\), a point in \(U\)

Try to apply \(\Phi\) ... Not defined since only many pts in \(\gamma_2^{-1}(P_2)\).

if \(\psi = \pi/2\), then at north pole regardless of \(\psi\) not bijective.

Unit sphere with north, south poles \(\{(\theta, \psi) \mapsto \begin{pmatrix} \cos \theta \cos \psi \\ \sin \theta \cos \psi \\ \sin \psi \end{pmatrix}\rangle\)

End up taking

\[U' = U \setminus (X_u \cup \gamma_2^{-1}(X_\psi))\]

\[V' = V \setminus (X_v \cup \gamma_2^{-1}(X_\psi))\]

\[\Phi: U' \rightarrow V' \text{ is ok.}\]
Examples that are fully computable are rare. Often get functions under square root with no elementary antiderivative - have to resort to numerical integration. Simple example: surface \((x, y) \mapsto (x, y, f(x, y))\)

\[ \mathbf{Dy}(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial}{\partial x} f & \frac{\partial}{\partial y} f \end{pmatrix} \]

\[ \mathbf{Dy}^T \mathbf{Dy} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \]

As a surface in \(\mathbb{R}^3\), so parametrization \(\gamma: \mathbb{R}^2 \to \mathbb{R}^3\)

then \(\det(\mathbf{Dy}^T \mathbf{Dy})^{1/2}\)

can be rewritten as

\[ \left| \begin{array}{cc} \frac{\partial}{\partial x} \gamma & \frac{\partial}{\partial y} \gamma \\ \frac{\partial}{\partial y} \gamma & \frac{\partial}{\partial x} \gamma \end{array} \right|^{1/2} \]

\(\times, y\): param. vars.

\(\) (or any other names for param. vars. \(x, y\))

might want to consider...

\(\) cross product

produces vector normal to the plane spanned by \(\frac{\partial}{\partial x} \gamma, \frac{\partial}{\partial y} \gamma\)

whose area = area of \(\gamma\)-region

bounded by these vectors.

Taking determinant: left with

\[ 1 + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \]

take square root.

\[ \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \]

not likely to have simple antiderivative.

Table of integrals only has quadratic functions under radical

i.e. \(\frac{df}{dx}, \frac{df}{dy}\) need to be linear.