On Monday, discussing volumes of discs on manifolds - particularly surfaces.

Set up a volume integral, but had two problems.

1) Remember how to find Taylor polynomials at $p \in M$.

2) Find open set $U$ to parameterize $D_r(p)$ on $M$: $\gamma: U \to D_r(p)$.

Almost done with (1) - remember how to find 2nd degree terms in Taylor expansion.

Along the way, defined curvature of 1-manifold at point $p$:

if $p$ locally expressible as $(f(x))$ in $R^2$, then curvature at $p$ is

$$\frac{f''(a)}{\sqrt{1 + f'(a)^2}}.$$ (in best coordinates $x, T$, it is just $\left| g''(0) \right|$ if $T = g'(x)$)

Wanted to move on to surfaces:

Again, we can pick best coordinates at $p$ for 2-manifold in $R^3$:

locally like $\begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$.

Pick new coords $\bar{x}, \bar{T}, \bar{B}$ so that $p$ is at $(0,0,0)$ in new coords. Then tangent plane to $p$ is $\bar{x}, \bar{T}$-plane with $\bar{B}$ axis normal to this plane.

Then in these best coords, we have

near $(0,0)$, $\begin{pmatrix} \bar{x} \\ \bar{T} \\ \bar{B} \end{pmatrix}$ with

Taylor expansion starting in degree 2 again:

write it as $\frac{1}{2} (A_{2,0} \bar{x}^2 + 2A_{1,1} \bar{x} \bar{T} + A_{0,2} \bar{T}^2) + \text{higher order terms}$.

studied these quadratic forms in optimization section. - wrote as sum of (or difference of) squares, in yet another change of coords.
In these coordinates, we have two notions of curvature (generalizing the one from 1-dimensional manifolds before).

You can think of this quadratic form \( A_{2,0} \bar{x}^2 + 2A_{1,1} \bar{x} \bar{y} + A_{0,2} \bar{y}^2 \)

\[
\begin{pmatrix} \bar{x} & \bar{y} \end{pmatrix} \begin{pmatrix} 2f/\partial \bar{x}^2 & 2f/\partial \bar{x} \partial \bar{y} \\ 2f/\partial \bar{y} \partial \bar{x} & 2f/\partial \bar{y}^2 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}
\]

\[
\text{symmetric matrix, \textit{Hessian}}
\]

hence \( 2A_{1,1} \) is natural to use basic invariants under conjugation action by invertible matrices: \( \det(A), \quad \text{tr}(A) \).

Mean curvature : \( \frac{1}{2} \ \text{tr}(A) = \frac{1}{2} (A_{2,0} + A_{0,2}) =: \mathcal{H}(p) \)

Gauss curvature : \( A_{2,0} A_{0,2} - A_{1,1}^2 \ = \ det(A) \ = \ K(p) \).

\[ \text{Gauss' theorem:} \quad \text{Area}(D_r(p)) = \pi r^2 - \frac{K(p)}{12} r^4 + \text{higher order terms in } r. \]

Problem : How do we compute \( K(p) \)? Need surface \( M \) to be in best coordinates \( \bar{x}, \bar{y}, \bar{z} \). Translation part is easy, just subtract.

Now we assume \( p \) is at \( (0,0) \) in \( \mathbb{R}^3 \).

\[ \bar{z} = f(x,y) \text{ has Taylor expansion} \ a_1 x + a_2 y + \frac{1}{2} (a_{2,0} x^2 + 2a_{1,1} xy + a_{0,2} y^2) \]

We win (i.e. are in best coordinates) if \( a_1, a_2 = 0 \).

Measure this with positive \( \pm c = \sqrt{a_1^2 + a_2^2} \).
then it turns out that \( K(p) = a_{2,0} a_{0,2} - a_{1,1}^2 \)

\[
\frac{a_{2,0} a_{0,2} - a_{1,1}^2}{(1 + c^2)^2}
\]

(not so different from the case of plane curves, going from best coords.)

Example: paraboloid \( z = x^2 + y^2 \). Find curvature at point \((a, b, a^2 + b^2)\).

Write \( X = x - a \)
\( Y = y - b \)
in \( X, Y, Z \) coords:
\[
Z = Z - (a^2 + b^2)
\]
\[
z + a^2 + b^2 = (x + a)^2 + (y + b)^2
\]

From this we compute:
\[
c = \sqrt{4(a^2 + b^2)}
\]
\[
c^2 = 4a^2 + 4b^2
\]
\[
K \left( \frac{a}{a^2 + b^2} \right) = \frac{4}{(1 + 4a^2 + 4b^2)^2}
\]

\( K \) at origin is 4.

this is a max. Away from origin, surface gets flatter.

In general, we can do any function of \( x, y \) making similar substitutions.

2nd Example: \( x^2 + y^3 + xy^2z - 3 = 0 \) at \((1, 1, 1)\). \( f : \mathbb{R}^3 \to \mathbb{R} \)

\[
\begin{bmatrix}
2x + y^2 - 3 & 3y^2 - 1 & 3xy^2
\end{bmatrix}
\]
\[
\begin{bmatrix}
2 & 3 & 3
\end{bmatrix}
\]
Jacobian:

\( 3xy^2 \bigg|_{(1, 1, 1)} = 3 \neq 0 \) so can write \( z \) as function of \( xy \). Call it \( f \)

Now solve for \( f \left( \frac{1 + x}{1 + y} \right) = 1 \)
\[ a_{110} = -1 \quad a_{011} = -\frac{4}{3} \]
\[ a_{210} = -\frac{2}{3} \quad a_{012} = -\frac{4}{9} \quad a_{111} = -\frac{2}{3} \]

So \[ c = \sqrt{(-1)^2 + \left(-\frac{4}{3}\right)^2} = \sqrt{\frac{25}{9}} \quad \text{and} \quad c^2 = \frac{25}{9}. \]

\[ a_{210} a_{012} - a_{011}^2 = \frac{52}{27} - \frac{4}{9} = \frac{40}{27}. \]

\[ \frac{\frac{40}{27}}{\left(1 + \frac{25}{9}\right)^2} = \frac{\frac{40}{27}}{\left(\frac{34}{9}\right)^2} = \frac{40}{34^2} = \frac{3}{17^2} \]

\[ = 0.1038... \]