Gauss' theorem

\[ \text{Area} (D_r(\mathbf{p})) = \pi r^2 - \frac{\pi K(\mathbf{p})}{12} r^4 + \text{higher order terms in } r. \]

Plan: Write parametrization

\[ \gamma : (\rho, \theta) \rightarrow \left( \begin{array}{l} \rho \cos \theta \\ \rho \sin \theta \\ f(\rho \cos \theta) \\ f(\rho \sin \theta) \end{array} \right) \]

(move \( \mathbf{p} \) to \( \mathbf{q} \) with simple initial change of coords)

\[ U \rightarrow D_r(\mathbf{q}) \]

Replace \( f \) with its Taylor poly. in best coordinates; choose span of tangent plane so that quad. terms are diagonal since symmetric matrices are always diagonalizable.

Revised result: Given \( f \), we have

\[ \gamma : (\rho, \theta) \rightarrow \left( \begin{array}{l} \rho \cos \theta \\ \rho \sin \theta \\ \frac{1}{2} (\rho^2 a \cdot \cos^2 \theta + \rho^2 b \cdot \sin^2 \theta) \\ + \text{higher order terms} \end{array} \right) \]

Show that

\[ \text{Area} (D_r(\mathbf{0})) \text{ def } = \int_U \left| \sqrt{\det[D\gamma(\rho, \theta) \times D\gamma(\rho, \theta)]} \right| d\rho d\theta \]

\[ = \pi r^2 - \frac{ab\pi}{12} r^4 + o(r^4) \]

Hard problem remains: Find open set \( U \) s.t.

\[ \gamma(U) = D_r(\mathbf{0}). \]

(or slightly less bad: \( \gamma(U) \) closely approximates \( D_r(\mathbf{0}) \) since answer only approximate)
In order to find/characterize the boundary of \( D_r(0) \), only need to keep shortest paths from 0 to a point \( x \) paths from 0 to \( x \) are given as functions \( \theta = h(\rho) \) some \( h \).

Path: \[ S(\rho) = \begin{pmatrix} \rho \cos h(\rho) \\ \rho \sin h(\rho) \end{pmatrix} \]

in \( x \)-\( y \) plane, with path lifted to surface

\[ \tilde{S}(\rho) = \begin{pmatrix} \rho \cos h(\rho) \\ \rho \sin h(\rho) \\ f(S(\rho)) \end{pmatrix} \]

Proposition: For \( r \) suff. small, the lift of the straight line path

\[ \tilde{\delta}(\rho) = \begin{pmatrix} \delta \rho(\rho) \\ f(\delta \rho(\rho)) \end{pmatrix} \]

is shorter than lift of any other path to

\[ \tilde{S}(\rho) \]

if, in Taylor expansion of \( S(\rho) \),

\[ S(\rho) = \theta_0 + K \rho + \frac{m}{2} \rho^2 + \cdots \]

with \( K \neq 0 \).

Guess: straight line paths in \( \mathbb{R}^2 \) lift to shorter paths in \( S \).

Book calls straight line path

\[ \delta \rho(\rho) = \begin{pmatrix} \rho \cos h(\rho) \\ \rho \sin h(\rho) \end{pmatrix} \]
pf of proposition (sketch) - Calculate approximate arc length for \( \tilde{S}(p) \) using Taylor expansion for \( h(p) \). (\( \tilde{\delta}_r(p) \) is special case with \( r = 0 \).

Arc length \( (\tilde{S}(p), p \in [0,r]) = \int_0^r \sqrt{|\tilde{S}'(p)|^2 + |D_\tilde{h}(S(p)) S'(p)|^2} \, dp. \)

\[
|S'(p)|^2 = (\cos h(p) - p \sin h(p) \cdot h'(p))^2 + (\sin h(p) + p \cos h(p) \cdot h'(p))^2
\]

\[
= 1 + p^2 \cdot h'(p)^2 = 1 + k^2 p^2 + o(p^2) \quad \text{as } r \to 0.
\]

Similar game for second term. Use Taylor poly of \( h(p) \) and for \( f \) in \( Df \).

leaves us with integrand \( \sqrt{1 + (\ldots)} \) but \( \sqrt{1 + x} = 1 + \frac{x}{2} + o(x) \)

Small if \( p \) small so substitute.

Length \( (\tilde{S}(p), p \in [0,r]) = r + \frac{r^3}{6} \left( k^2 + (a \cos^2 \Theta_0 + b \sin^2 \Theta_0)^2 \right) \)

\[\text{this expression gets smaller if } k > 0 \quad \text{(in } (*) \text{)}\]

Now to finish problem, we see lifts of straight line paths \( \tilde{\delta}_r(p) \) have arc length \( (\tilde{\delta}_r(p), p \in [0,r]) = r + \frac{r^3}{6} \) \,(const.) + o(r^3) \quad \text{from } k = 0 \quad \text{(in } (*) \text{)}\)

Given length \( r \) in any direction, get lifted path \( r + \frac{r^3}{6} \) \,(const.)

Ask reverse question: given piece \( \tilde{\delta}_r(p) \) path upstairs up to \( o(r^3) \)

What is \( \tilde{\delta}_r(p) \) downstairs? Use inverse function. Don't know in general

but \( r + \text{(const.)} r^3 \) has inverse \( r - \text{ same Const. } r^3 \).