Exam Wednesday.

Covers Section 4.11 - Lebesgue integration + Ch. 5 (1-4) on integration on manifolds.

All exams are cumulative, but really only in Lebesgue int.

That we require some facts from Riemann integration.

Major topics in Lebesgue integration:

- Definition of Lebesgue integral
- Proving that functions are $L$-integrable
- Easy properties ("Some elementary properties of Lebesgue integral")
- Career-making hard theorems
- Differentiation under integral sign (Bringing limits inside)

Definition: Write $f = \sum_{k=1}^{\infty} f_k$ for $f_k: R$-integrable. In particular, bounded with (bounded supp.) also: cont on set of wear 0.

(Note need $\sum_{k=1}^{\infty} \int_{R^n} |f_k(x)| \, dx < \infty$)

Ensures sum is convergent a.e.

Set $\int_{R^n} f(x) \, dx = \sum_{k=1}^{\infty} \int_{R^n} f_k(x) \, dx \, dx$.

Hard part: Find these functions $f_k$, R-integrable.
Prototypical examples:

(0) $R$-integrable

(1) unbounded support (but decays fast enough)
   e.g. $R^2 - B_1(0)$ with $|x|^p$ for various $p$. 
   find $f_k$ supported on larger and larger domains as $f_k$ function of $k$.

(2) unbounded values
   e.g. $|x|^p$ with unit ball and $p < 0$.

Non-examples: oscillating functions for which $|f|$ not integrable.
   e.g. $\sin \frac{1}{x}$

How do we show that $\int_{R^2} \frac{1}{x} \, dx$ is not Lebesgue integrable? Can't take improper integral of it,
   so we're suspicious, but don't know how to connect
   the improper $\int$ Lebesgue integr.

Use fancier property.
   e.g. $\int_0^1 \frac{1}{x} \, dx$ send $x \mapsto x^2$
      (nice map on $(0,1) \to (0,1)$)

$= \int_0^1 \frac{1}{x^2} \left| 2 u \right| \, du = \int_0^1 \frac{2}{u} \, du = 2 \int_0^1 \frac{1}{x} \, dx$
Example: function has a singularity (place where blows up) at origin.

Integral is over compact region. E.g. unit ball.

In \( \mathbb{R}^n \):

There might try chopping up unit disk into annuli.

Q. Does width of annuli matter?

If width \( \left[ \frac{1}{k+1}, \frac{1}{k} \right] \), \( k > 1 \),

\[
\text{area} = \pi \left( \frac{1}{k} \right)^2 - \pi \left( \frac{1}{k+1} \right)^2
\]

Which functions can we handle?

Suppose \( f \) increasing toward 0, so \( f \) takes max at \( \frac{1}{k+1} \).

If we set \( f_k = f \) on interval \( \left[ \frac{1}{k+1}, \frac{1}{k} \right] \), then it is R-integrable if \( f \) is \( C^1 \) elsewhere.

Now have to examine whether

\[
\sum_{k=1}^\infty \int_{1/k+1}^{1/k} |f_k| \, d\nu < \infty ?
\]

Estimate integrals using max or actually try to compute using Fubini.

If we use the max, we get

\[
\int |f_k| \, d\nu < f \left( \frac{1}{k+1} \right) \cdot \left[ \pi \left( \frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \right]
\]

If we use \( \left[ 2^{-k+1}, 2^{-k} \right] \)

Get

\[
\pi \left( (2^{-k})^2 - (2^{-(k+1)})^2 \right) \leq \pi \cdot 2^{-2k}
\]

More accurate if we can integrate \( f \) over annulus, e.g. using polar coords.
Differentiation under integral sign.

Example: \( \int_0^\infty \cos(tx) e^{-x^2/2} \, dx = ? \)

First justify that differentiating under integral sign is possible.

Integral must exist for each fixed \( t \), + some version of dominated convergence theorem.

Then differentiating with respect to \( t \),

call integral above \( F(t) : \)

\[
\frac{d}{dt} F(t) = \int_0^\infty -x \cos(tx) e^{-x^2/2} \, dx
\]

now do integration by parts:

\[
u = \sin tx \quad dv = -x \cdot e^{-x^2/2} \\
du = t \cos tx \quad v = e^{-x^2/2}
\]

\[
\frac{d}{dt} F(t) = \left[ \sin tx e^{-x^2/2} \right]_0^\infty - t F(t)
\]

\[
\sim 0
\]

So \( F'(t) = -t F(t) \quad \Rightarrow \quad F(t) = C \cdot e^{-t^2/2} \)

solve for \( C \) by setting \( t = 0 \).

Then \( C = \int_0^\infty e^{-x^2/2} \, dx = \sqrt{\frac{\pi}{2}} \).