Last time: Orientations on manifolds \(\rightarrow\) continuously varying assignment of orientation to tangent space 

Goal:
- Assign orientation to manifold
- Find a parametrization to do integration
- Determine whether parametrization preserves orientation.

Big theorem: If param. preserves orientation, then integral of \(k\)-form on oriented \(k\)-manifold is independent of choice of such parametrization.

(Use same definition of integration as ever:
\[
\int_M \varphi = \int_U \varphi(\gamma(u))(\text{D}\gamma(u))(d^k u),
\]

How to assign orientation?

For curve, choose nowhere vanishing, continuously varying tangent vector

then choose \(\Omega_x^k(y) = \text{sgn} (\pm(x) \cdot y)\) \(\pm(x)\) for \(x \in \mathbb{R}\).

Example: unit circle \(x^2 + y^2 = 1\).

then \(DF = \begin{pmatrix} 2x \\ 2y \end{pmatrix}\) and \(T_x M = \ker(DF) = \langle (-y) \rangle\) span of this vector at \((x,y)\).

\((-y, x)\) non-vanishing on unit circle
(or any circle of radius \(R > 0\))

so defines orientation \(\text{sgn} ((-y,x) \cdot y)\)
for surface, find continuously varying vector $\mathbf{\eta}$ as function of $x \in M$, write $\mathbf{\eta}(x)$ NOT in tangent space.

Choose $\Omega^{\frac{\mathbf{\eta}(x)}{||\mathbf{\eta}(x)||^2}}(x, y, z) = \text{sgn} \left( \det \left( \begin{array}{ccc} \mathbf{v}_1(x) & \mathbf{v}_2(x) \\ \mathbf{v}_3(x) & \mathbf{v}_4(x) \end{array} \right) \right)$

$\Omega(\mathbf{\eta}(x))$ live in tangent space
together they define honest 3-parallelogram so has non-zero 3-volume.

Example: Pick $[DF]^\top$ if $\text{DF}(x)$ non-vanishing for all $x \in M$

$[DF]^\top$ (3x1 vector since $F: \mathbb{R}^3 \to \mathbb{R}^1$)

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(in higher dimensions $DF^\top: \mathbb{R}^n \to \mathbb{R}^{n-k}$, needs to be surjective)

at each $x$

to give non-deg $n$-parallelogram.

$\text{(3) Checking if parametrization preserves orientation:}$

Say $\gamma$ is orientation preserving if $\Omega(\gamma(\mathbf{\eta}(u))) = +1$ $\forall u \in U - X$ ($\gamma$: relaxed param

Example: unit circle $\gamma(\mathbf{u}) = \begin{bmatrix} \sin \mathbf{u} \\ \cos \mathbf{u} \end{bmatrix}$

Does it preserve orientation of

$\Omega_{\frac{\gamma}{\gamma}}(x) = \text{sgn} \left( \frac{\gamma(x) \cdot \mathbf{v}}{||\gamma(x)||} \right)$

Compute $\pm (\gamma(\mathbf{u})) \cdot \gamma(\mathbf{u}) = -1$. No!
More serious issue: spherical coords for unit sphere in $\mathbb{R}^3$

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} \cos \phi \cos \theta \\ \cos \phi \sin \theta \\ \sin \phi \end{pmatrix} \quad \theta \in [0, \pi] \quad \phi \in [-\pi, \pi]$$

Choose $\Omega$: $\det \begin{bmatrix} n(x_1) & n(x_2) & n(x_3) \end{bmatrix}$ with $n(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

then you can check $\det \begin{bmatrix} n(\chi(\theta, \phi)), D\chi(\theta, \phi), D^2\chi(\theta, \phi) \end{bmatrix}$

$= -\cos \phi$

so doesn't respect orientation, since $\cos \phi$ oscillating.

How could this be? Parametrizations supposed to give orientations.

Problem: spherical coords are relaxed parametrization. (At north/south poles where $\cos \phi = \pm \pi/2$, get all $\theta$ mapping to same point.)

(i.e. $\cos \phi = 0$)