Last time, outlined 3 part plan:

1. try to find an orientation
2. try to find a parametrization (relaxed)
3. determine if parametrization is orientation preserving.

Big theorem (later today): If we define

$$
\int_M \Phi \equiv \int_\mathcal{X} \Psi(\gamma(u))(D\delta(u)) \|d^k u\| \quad \text{for any relaxed } \gamma: \mathcal{X} \to \mathcal{M}
$$

(Better to write out first as equality of two integrals w.r.t. relaxed param.) (then this is well-defined)

Suppose true for now and do some examples.

\( M = \text{plane in } \mathbb{R}^3 \). e.g. \( 2x + 3y - z = 6 \) with \( x, y \geq 0 \) \( z \leq 0 \).

Find orientation — Pick nowhere vanishing vector at each point not in tangent plane.

Obvious choice: constant normal vector \((2, 3, -1)\),

\( \Omega : (x, v_1, v_2) \mapsto \begin{pmatrix} x \\ \mathbf{s}\,\mathbf{g}\,\mathbf{n} \end{pmatrix} \begin{vmatrix} 2 & 1 & 1 \\ 3 & v_1 & v_2 \\ -1 & 0 & 1 \end{vmatrix} \)

Find parametrization: \( z = 2x + 3y - 6 \) so

\( \gamma : (x, y) \mapsto (x, y, 2x + 3y - 6) \).

\( D\delta(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \) for all \( x, y \).

\( \Omega : (x, v_1, v_2) = \begin{pmatrix} x \\ \mathbf{s}\,\mathbf{g}\,\mathbf{n} \end{pmatrix} \begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ -1 & 2 & 3 \end{vmatrix} \)
\[
\text{det } \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ -1 & 2 & 2 \end{pmatrix} = 2 \cdot (-2) - 1 \cdot 6 = -10. \text{ get } -1 \text{ at all points!}
\]

Orientation reversing! Could change vector to \((-2, -3, 1)\) defining orientation.

Could change parametrization.

Consider writing map like \(A: \mathbb{R}^k \to \mathbb{R}^k\)

\[
A(x_1, x_2, \ldots, x_k) = (-x_1, x_2, \ldots, x_k)
\]

\[
\det A = \begin{vmatrix} -1 \\ \vdots \\ -1 \end{vmatrix} = -1.
\]

Map \(A^{-1}(U) \to M\) instead of \(U \to M\).

or just take final integral to be negative of orientation-reversing one.

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Now can integrate any two-form field. \(\varphi = x^2 \; dy \wedge dz\).

\[
\int \varphi = \iint_P x^2 \; dy \wedge dz.
\]

Triangle in \(xy\)-plane

\[
= \iint_{\text{Triangle}} x^2 \; dy \wedge dz \cdot \left( \frac{x}{2x+3y-6} \right) \cdot \left( \text{D}Y(\vec{x},\vec{y}) \right) \cdot d(\vec{x},\vec{y})
\]

\[
= \iint_{\text{Triangle}} x^2 \; dy \wedge dz \cdot \left( \frac{y}{2x+3y-6} \right) \cdot \text{det of submatrix} - x^2 \cdot \text{det of submatrix} = -2.
\]

No harder for arbitrary graph \(z = f(x, y)\). Remember how to define normal vector.