Defined $d_{M}X$ for $X$: compact subset of $\mathbb{R}$-manifold $M$.

and smooth boundary $d_{M}^{s}X$ (determined by odd smooth function $g: U \subseteq \mathbb{R}^{k} \to \mathbb{R}$ for each point $x \in d_{M}X$.)

So sets $X$ we wish to consider have:

\[
\text{vol}_{k-1}(d_{M}^{s}X) < \infty
\]

\[
\text{vol}_{k-1}(d_{M}^{\text{n.s.}}X) = 0. \quad d_{M}^{\text{n.s.}}X: \text{non-smooth boundary} = d_{M}X \setminus d_{M}^{s}X.
\]

**FACTS:**

1. $\text{vol}_{k}(X) < \infty$ if $X$: piece with boundary.

2. If $X$ is piece with boundary, then $g(X)$ is piece with boundary if $g: A \xi + \xi$ is linear map with $A$: invertible $n \times n$ matrix.

**Running example in section:** $k$: parallelogram anchored at $\xi \in \mathbb{R}^{n}$, $P$,

spanned by $v_{1}, \ldots, v_{k}$.

Think of it as sitting in $k$: hyperplane spanned by $v_{1}, \ldots, v_{k}$ anchored at $\xi$.

this is our $M$.

To find $F: \mathbb{R}^{n} \to \mathbb{R}^{n-k}$ defining $M$, just pick $A$ with $\ker(A) = \langle v_{1}, \ldots, v_{k} \rangle$

define $F(y) = A(y) - A(\xi)$

fixed rooted point of $k$: parallelogram.
Our function \( g \), locally defining boundary, is the linear function

\[ \alpha_i : \mathbb{R}^n \to \mathbb{R} \text{ such that } \ker [A_i] = \text{span (k-1 vectors spanning boundary k-1 parallelogram)} \]

then additional inequality is

\[ \alpha_i(x) \leq \alpha_i(y) \leq \alpha_i(x+y) \]

So \( g(y) = \alpha_i(y) - \alpha_i(x) \) or \( \alpha_i(y) = \alpha_i(x+y) \).

The non-smooth point is intersection of two k-1 planes, a k-2 plane which has k-1-volume equal to 0.

**How do we orient boundary?** (Need this to do oriented integration on \( \partial_M X \) required in Stokes' theorem)

**Answer:** Not so bad. Just use orientation on any open set \( U \subseteq M \) that contains our compact set \( X \).

(You even manifolds w/o global orientation are ok as long as we find orientation on \( U \supset X \)).

**Idea:** Have orientation on \( X \subset M \) k-manifold with \( T_X M \) having basis spanned by k vectors in \( \mathbb{R}^n \).

\( T_X \partial_M X \) will have k-1 vectors. Complete this to a basis of \( T_X M \) using one more vector.

\[ (v_1, \ldots, v_{k-1}) \mapsto \Omega^M_X (v_{\text{special}}, v_1, \ldots, v_{k-1}) \]

What is \( v_{\text{special}} \)? Use our extra condition \( g(x) = 0 \) to pick consistent choice of \( v_{\text{special}} \).
If, given \( v \in T_x M \setminus T_x J^*_M X \), we have

\[
[Dg(x)] \cdot v > 0, \quad \text{this means } v \text{ points into domain } X
\]

\[
[Dg(x)] \cdot v < 0, \quad \text{then } v \text{ points outward from domain } X.
\]

Choose \( \nu_{\text{out}} \) to be \( \nu_{\text{special}} \) — that is, for each \( x \), pick \( \nu_{\text{out}} \) in \( T_x M \).

This is our orientation.

**Example:** \( M = \mathbb{R}^2 \), \( X : \text{compact} \)

\[
\Omega^2_x (\nu) := \text{sgn } \det (\nu_{\text{out}}, \nu)
\]

with \( \nu \in T_x \mathbb{R}^2 \)

\( \nu_{\text{out}} \) first means move counterclockwise from \( \nu_{\text{out}} \) to \( \nu \)

in pos. orientation.

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**Example 2:** Surface \( M \subset \mathbb{R}^3 \) with orientation given by normal vector \( n(x) \).

with

\[
\Omega^2_M (\nu_1, \nu_2) := \text{sgn } \det [n(x), \nu_1, \nu_2]
\]

then set \( \nu_1 = \nu_{\text{out}} \) get

\[
\Omega^2_x (\nu) := \text{sgn } \det [n(x), \nu_{\text{out}}, \nu]
\]

Can do 3-manifolds as open charts of \( \mathbb{R}^3 \) bounded by surfaces as well.

Or 1-manifolds whose boundary is points, to recover FTC.