Stokes' theorem: \[ \int_X \Phi = \int_{\partial M} \Phi \] \[ \int_X \Phi = \int_{\partial M} \Phi \] Need to understand \( \partial M \) "good" \( \partial M X \) \( \Phi \) \( (k-1) \) form (field) \( \Phi \) on \( X \) "exterior derivative" \( \Phi \).

Should generalize FTC: In our language, \( X = [a,b] \subseteq \mathbb{R} \),

with \( \partial M X = \{a,b\} \) (orientation on \( b \) is \( + \), on \( a \) is \( - \))

and \( \Phi \) is 0-form, aka function, call it \( F \).

Then \( \int_X F = F(b) - F(a) \).

\( \int_{\partial M X} F \) \( [a,b] \) (exterior deriv. on \( F \)

is derivative).

Also want Stokes' theorem to be true,

so how to define \( d\Phi \) so that it holds?

Along shared boundary have opposite orientation:

\[ \text{define } d\Phi (x)(v_1, \ldots, v_k) = \lim_{h \to 0} \frac{1}{h} \int_{\partial P_x(hv_1, \ldots, hv_k)} \Phi \]

\( \partial P_x(hv_1, \ldots, hv_k) \)

boundary of a \( k\)-cell is on almost everywhere \((k-1)-manifold\).

Concret this in a way makes Stokes' theorem believable.

Trouble is: can we compute \( d\Phi \) for \( \Phi \) a \( k \)-form with \( k > 0 \).

Amazingly, yes!
Bry theorem on computing $d\psi$, $\psi$ a $k$-form.

1. The limit exists if $\psi$ is nice:
   \[ \psi = \sum_{i_1, \ldots, i_k} a_{i_1, \ldots, i_k}(x) \, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \]
   over $C^2$-functions on nbhd $U$.

2. Linearity: $d(a\psi + b\psi) = ad\psi + bd\psi$.

3. Constants: $\psi = \sum_{i_1, \ldots, i_k} c_{i_1, \ldots, i_k} \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ (constant form)
   then $d\psi = 0$.

   (Special case of 5, so slippery)

4. $df$, $f$ function (aka 0-form):
   \[ df = \left[ Df \right] = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right) \, dx_i \]

5. Given $f$,
   \[ d\left( f \, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \right) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \]

All you need:

- $d \left( e^x \, y \, dx \wedge dz + 2x \, dy \wedge dz \right)$

  - $d \left( e^x \, y \, dx \wedge dz \right) + d \left( 2x \, dy \wedge dz \right)$

  - $d \left( e^x \, y \right) \wedge dx \wedge dz + \frac{d(2x)}{2} \wedge dy \wedge dz$
\[ d(e^x y) = D_x (e^x y) \, dx + D_y (e^x y) \, dy + D_z (e^x y) \, dz \]
\[ e^y \, dx + e^x \, dy = 0 \]

so \[ d(e^x y) \wedge dx \wedge dz = (e^x y \, dx + e^x \, dy) \wedge dx \wedge dz \]
\[ = e^y \, dx \wedge dx \wedge dz + e^x \, dy \wedge dx \wedge dz \]
\[ = 0 \quad \text{(earlier property of wedge product)} \]

Two useful corollaries (both of which follow from the theorem by computation):

- \[ -e^x \, dt \]

Thm 6.7.7 in H-H: \( \psi \) nice, then \( d(d\psi) = 0 \).

Thm 6.7.9 in H-H: \( d(\psi \wedge \psi) = d\psi \wedge \psi + (-1)^k \psi \wedge d\psi \).

(\( \psi \), \( \psi \) nice, \( \psi \) a \( k \)-form, \( \psi \) an \( l \)-form)
0-form field \( f \) so \( df \) is 1-form.

\[
\begin{align*}
\text{df} (x) (v) & \overset{\text{def}}{=} \lim_{h \to 0} \frac{1}{h} \int_{\partial P_x (hv)} f = \lim_{h \to 0} \frac{f (x + hv) - f (x)}{h} \\
& = [Df (x)] v \\
\text{i.e.} & = (D_1 f) v_1 + \ldots + (D_n f) v_n
\end{align*}
\]

Harder:

\[
d (f dx_{i_1} \wedge \ldots \wedge dx_{i_n}) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_n}.
\]

Translate form to origin to find value at any particular pt.

Use Taylor expansion for \( f \):

\[
f = f (0) + \sum \frac{D_i f (0)}{i!} x_i + \ldots + \frac{D_n f (0)}{n!} x_n + \text{remainder}.
\]

Show that linear term is what contributes to limit.

Integrate over faces of parallelograms 2 \((k+1)\)-of them.

Parametrize them and integrate:

pairs with one vector fixed at \( 0 \cdot v_i \) or \( h \cdot v_i \)

other vectors free to roam: \((t_1, \ldots, t_k) \mapsto 0 \cdot v_i + t_1 v_1 + \ldots + t_k v_k + t_{k+1} \sum v_{k+1}

Consider orientation.

Pairs have cancelling constant terms. Work out linear term.

\( t_j \in [0, 1] \)
linear forms of opposing faces:

\[ \int_{[0,1]^k} \left( \varphi_i(t) \left( \gamma_{1,i}(t) - \gamma_{0,i}(t) \right) \right) \, dx_{i_1} \wedge \ldots \wedge dx_{i_k} (v_1, \ldots, \hat{v_i}, \ldots, v_{k+1}) \]

\[ \text{volume of opposing faces} \]

\[ \left[ Df(0) \right] (hv_i + \gamma_{0,i}(t)) - \left[ Df(0) \right] \gamma_{0,i}(t) \]

\[ = \frac{\partial}{\partial t} \left[ Df(0) \right] v_i \]

So, summing over all:

\[ = \sum_{i=1}^{k+1} (-1)^{i-1} \frac{h}{h^{k+1}} \left[ Df(0) \right] v_i \left( dx_{i_1} \wedge \ldots \wedge dx_{i_k} \right) (v_1, \ldots, \hat{v_i}, \ldots, v_{k+1}) \]

\[ = \text{wedge prod formula} \]

Easier about this: normally \( DX_{i_1} \ldots DX_{i_0} \) inserted into form change depending on \( t \). Not here since

\[ \text{program is linear.} \]