On Friday, defined exterior derivative as flux: \( \mathcal{L} : p-1 \text{- form} \)

\[
d\psi(x)(v_1, \ldots, v_k) = \lim_{h \to 0} \frac{1}{h^k} \int_{\partial P_x(hv_1, \ldots, hv_k)} \psi
\]

so that Stokes' theorem (at least approximately) appeared true.

Owes: show \( d\psi \) compatible.

Trying to show \( \partial \left( f \, dx_1 \wedge \ldots \wedge dx_k \right) = df \wedge dx_1 \wedge \ldots \wedge dx_k \).

Plan: use definition, pairing opposite sides of \( \partial P_x \) (which have opposite orientation) and Taylor expansion of \( f \) (with \( x \) moved to 0).

Key pts: Derivative same on both sides in vars \( \neq i \).

Linear terms of \( f \) (with orientation on each boundary)

\[
\left[ Df(0) \right] (hv_i + \gamma_{0,ii}(t)) - \left[ Df(0) \right] (\gamma_{0,ii}(t))
\]

\[
= h \cdot \left[ Df(0) \right] v_i.
\]

so plugging into definition of exterior deriv: \( \frac{\partial (v_j's)}{\partial t_i} \)

\[
\lim_{h \to 0} \frac{1}{h^k} \int_{\partial P_x(t_1v_1 + \ldots + hv_i + \ldots + t_{k-1}v_k)} \psi
\]

\[
\left[ Df(0) \right] \mathbf{v}_i
\]

\[
\text{translation of } \partial P_x
\]
Return to connection to physics. Favorite forms in $\mathbb{R}^3$.

Work form $W_F(x)(x_1, x_2) = F(x_1) \cdot x_1 \cdot x_2 \cdot x_1$ (1-form)

Flux form $\Phi_F(x)(x, w) = \text{det} \begin{bmatrix} F(x_1) & x_1 & w \\ x_2 & x_2 & 0 \\ x_3 & x_3 & 0 \end{bmatrix}$

($+$ 0-forms (functions) and 3-forms (volume via det.)

integral: evaluation integral: "mass" - weighted sum of volumes.

Natural question: What is effect of exterior derivative on favorite forms?

Given 0-form $f$, apply $df$. It's a 1-form and all 1-forms are $W_g$ for some $g$.

So what is $g$?

$\text{Ans: } df = W_{\nabla f}$ where $\nabla f =$ "gradient of $f$" or write $\text{grad}(f) := \begin{bmatrix} D_1f \\ D_2f \\ D_3f \end{bmatrix}$

(equally true in $\mathbb{R}^n$, going from 0-forms to 1-forms)

Given 2-form $\Phi_F$, then $d\Phi_F =$ mult. of det (call it $M_F$

for "mass form")

$\Theta = \nabla \cdot \Phi$

$\nabla \cdot \Phi = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$

$= D_1F_1 + D_2F_2 + D_3F_3$

"div($F$)"

(do example with genuine function...)

In $\mathbb{R}^n$, physical intuition for 0, 1, n-1, n forms.

In $\mathbb{R}^3$, happens that $n-1 = 2$, so can try to express $dW_F$ as flux form $\Phi_G$.
So what is $G$ s.t. $dW_f = \nabla \cdot \vec{F}$? $G = \nabla \times \vec{F}$

"curl of $\vec{F}$"

$$
\begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix} \times 
\begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix} = 
\begin{bmatrix}
D_2 F_3 - D_3 F_2 \\
D_3 F_1 - D_1 F_3 \\
D_1 F_2 - D_2 F_1
\end{bmatrix}
$$

An example is a little singular since only in $\mathbb{R}^3$ that we can get from work to flux.

What is physical intuition behind div, grad, curl?

Gradient form $W_{Df}(x)(y) = [Df(x)] \cdot y$

This is directional derivative of $f$ in direction of $y$.

It is fastest when $y$ points in same direction as $[Df(x)]$ (biggest)

so $[Df(x)]$ = "grad $f(x)$" points in direction of fastest increase.

What about div? Just fall back on definition of $d$ as flux.

$$(\text{div } F) \cdot dx \wedge dy \wedge dz = \frac{d}{dG}$$

Flux in small box around $x$

Curl is more painful to explain...

Other useful properties we didn't mention

$$d(\omega) = 0$$ $\omega$ k-form, class $C^0$

$$d(\varphi \wedge \psi) = 0$$ $\varphi \wedge \psi$
Definition: A vector field is called "rotation-free" if $\text{curl}(\mathbf{F}) = 0$.
and "incompressible" if $\text{div}(\mathbf{F}) = 0$.

check the following properties (consequences of fact that $\text{d}(\text{d} \Phi) = 0$
or just check directly...)

1. $\text{curl}(\text{grad}(f)) = 0$

2. $\text{div}(\text{curl}(\mathbf{F})) = 0$

Example: A magnetic field is always expressible as $\text{curl}(\mathbf{A})$
for some vector field $\mathbf{A}$. Thus, magnetic field is always
incompressible.