Definition: A vector field is called "rotation-free" if \( \text{curl}(\vec{F}) = 0 \) and "incompressible" if \( \text{div}(\vec{F}) = 0 \).

Check the following properties (consequences of fact that \( d(d\psi) = 0 \) or just check directly ...)

1. \( \text{curl}(\text{grad}(f)) = 0 \)
2. \( \text{div}(\text{curl}(\vec{F})) = 0 \)

Example: A magnetic field is always expressible as \( \text{curl}(\vec{A}) \) for some vector field \( \vec{A} \). Thus, magnetic field is always incompressible.
Examples of Stokes' theorem: \( X = M : k\)-manifold \( \subset \mathbb{R}^n \), \( \varphi \) : \( k-1 \)-form compact, "good" boundary

\[
\text{then } \int_X \varphi = \int_X d\varphi.
\]

Easy example: \( X = \) rectangle in \( \mathbb{R}^2 \). Then \( \int_X \varphi \), \( \varphi \) : 1-form

say with vertices

\((0,0), (a,0), (0,b), (a,b)\)

is painful as # of sides to boundary is \( 2n = 4 \). (worse in higher dimensions)

Pick \( \varphi : x \, dy - dx \)

\( d\varphi = d(x \, dy - dx) = dx \wedge dy - d(dx) = dx \wedge dy \)

so by Stokes' thm.

\[
\int_X \varphi = \int_X d\varphi = \int_X dx \wedge dy = a \cdot b.
\]

Harder example over cube in \( \mathbb{R}^3 \),

integrating 2-form over boundary of cube.
Even if want to integrate over \((k-1)\)-manifold not a boundary, still play tricks: e.g. want to integrate over \(C\), complete it to boundary of \(2\)-manifold using \(C'\), where \(C \cup C' = \partial X\)

Stokes' theorem gives
\[
\int_C \varphi = \int_X df
\]

So,
\[
\int_C \varphi = \int_C df - \int_{C'} \varphi.
\]

For example, if \(df = 0\) then \(\int_C \varphi = -\int_{C'} \varphi\)

\(C'\) any curve with same endpoints. \((C' \text{ nice enough})\)

\(\varphi = x \, dy + y \, dx\)

then \(df = dx \wedge dy + dy \wedge dx = 0\).

pick 0-form \(f\), then \(df\) is 1-form and \(d(df) = 0\).

\(f = xy\) then \(df = dx \cdot y + dy \cdot x\).

Earlier sketch of Stokes' theorem:
\[
\int_X df = \sum_{i=1}^N \int_{\partial P_i} df \\
\text{assume \(df\) constant on small \(P_i\)}
\]

\(\sum_{i=1}^N \int_{\partial P_i} df \approx \int_X df\)

definition of \(d\) as flux.

\(\int_X df\) orientations on boundaries cancel.

Works nicely if boundary of \(X\) is well-approximated by dyadic paving.