Semester as progression to Stokes' theorem: \( X \subseteq M \) compact, good boundary

\[ \int_X \varphi = \int_X d\varphi, \quad \varphi \text{ class } C^2\text{-form } (k-1 \text{ if on } X, \dim(M) = k) \]

1. Integration using Riemann sums
   (dyadic pavings of compactly supp. functions)
   with compact values.

3 key results:
   a) Integrability (iff discontinuous on a set of measure 0.)
   b) Fubini's thm (iterated integration)
   c) Change of vars. theorem

lots of hypotheses, on change of vars function \( \varphi \) 

1/2 step to Lebesgue integrability

Infinite sums of Riemann integrable functions.

4 key results:
   a) Well definedness
   b) Dominated convergence theorem
   c/d) Fubini's thm, change of vars.

\( f_k \to f \), and \( \exists L\text{-int. } F \text{ s.t. } |f_k(x)| \leq F(x) \text{ a.e.} \)

then \( f \) is integrable and equal to \( \lim \int_X f_k \).

(leads to diff. under integral e.g.)
2. Integration on manifolds.

- Two key topics:
  - Relaxed parameterizations
  - Curvature.

U ⊆ \mathbb{R}^n" with "bad place X"

\gamma(U) \supset M \supset \gamma(U - X)

\text{like before: one-one}
\begin{align*}
&\text{C^1 with Lipschitz deriv. (loc.)} \\
&\text{DY one-one \& well X}
\end{align*}

k-volume of X

\text{and } \gamma(X) \text{ a compact}

= 0 \text{ \& compact.}

\text{Lipschitz if function is } C^2
\text{ deriv}

\text{(even had way of producing Lipschitz ratio from second partials long time ago)}

\text{(error given by curvature)}

\text{Area (D}_t f) = \pi r^2 - \pi k(r) r^4

\text{+ o(r^4)}

\text{Make this our pf...}

(Each chapter has important)

"independence of param."

Using change of vars

but won't pick it each time

3. Oriented integration on manifolds. (~40% of test)

- Differential forms (evaluate them, take exterior deriv, simple properties)
  - Orientations (definition + key examples (use form to make orientation), consistent with param:)

\text{integrating k-forms over parameterized domain}
\begin{align*}
\Omega \quad \text{(D}_t \gamma(U)) = 1 & \quad U \in U - X \\
\gamma \quad U \longrightarrow M
\end{align*}
Example: \[ \int_C xy \, dy + x^3 \, dx \] where \( C \) is a semi-circular curve with radius \( R \).

Parametrize each piece. Or use Stokes' theorem in \( \mathbb{R}^2 \).

Take \( d ( f \, dx + g \, dy ) \) to make 2-form. In general, only \( dy + \theta \) survives

\[ dy + \theta = \sin \theta \, dx \wedge dx + \, dx \wedge dy \]

In \( \mathbb{R}^2 \), identity:

\[ \int_C f \, dx + g \, dy = \int_S (D_x g - D_y f) \, dx \wedge dy \]

Green's/Stokes' theorem \( \Rightarrow \)

\[ \int_C xy \, dy + x^3 \, dx = \int_S y \, dx \wedge dy \]

\[ S = \text{semi-circle} \]

\[ = \int_0^\pi \int_0^R \sin \theta \, r \, dr \, d\theta \]

\[ = \frac{R^3}{3} \left[ -\cos \theta \right]_0^\pi \]

\[ = \frac{2R^3}{3} \]

A little bit fast about checking that parametrization using polar coordinates was consistent with orientation.

\( \Omega (D \gamma (u)) = 1 \) \( \forall \ u \in \mathbb{R} \times (0, R) \).
One more example: Compute the flux through the unit cube of the vector field \( \mathbf{A}(x, y, z) = \begin{bmatrix} x y^2 + z \\ 4 y z - x z \\ x \cos y \end{bmatrix} \) with standard orientation on cube. (outward pointing normal)

Again, Stokes' theorem:

Nice to remember that flux form

\[
d: \mathbf{A}(x, y, z) \rightarrow \text{Mass form } M \frac{\text{div}(\mathbf{A})}{\nabla \mathbf{A}}
\]

2-form we get from \( \det \begin{bmatrix} F & V \\ W & 1 \end{bmatrix} \) then

Vector field assoc. to \( \mathbf{A} \):

\[
\int_{\text{C}} \mathbf{A} \cdot d\mathbf{r} = \int_{\text{C}} d\mathbf{A} = \int_{\text{C}} M \text{div}(\mathbf{A})
\]

(a divergence theorem)

Claim: \( \mathbf{F}(x, y, z) = \frac{\partial}{\partial x} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \)

\[
= F(x) \cdot N(x) \quad |d^2 x|
\]

unit normal vector

\[
\left( N(x) \left| \frac{\partial}{\partial x} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right| \right) \cdot (v, w)
\]

\[
= N(x) \cdot |v \times w|
\]

\[
= v \times w
\]

so just check that \( \mathbf{F}(x) \cdot (v \times w) = \det (F, v, w) \).

\[
\int_{0}^{1} \frac{1}{3} y^3 + 4 y z \left| \begin{array}{c} \frac{1}{3} + 4 z \\ dt \end{array} \right|
\]

\[
\int_{0}^{1} \frac{1}{3} + 4 z \left| \begin{array}{c} dt \end{array} \right|
\]

\[
\frac{7}{3}
\]