Numeric integration in 1-variable:

Discussed this briefly last semester when discussing interpolation of polynomials. Idea: Given function \( f \), model it by quadratic passing through three equally spaced points

\[
T: \mathbb{P}_2 \rightarrow \mathbb{R}^3 \quad \text{found matrix for } T, \quad \text{inverted}
\]

\[
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\
\end{bmatrix} = \begin{bmatrix}
p(0) \\ p(0) \\ p(1) \\
\end{bmatrix} \quad \text{leads to general formula:}
\]

\[
\begin{bmatrix}
\frac{y_1}{2} \\ \frac{y_2 - y_0}{2} \\ \frac{y_0 - 2y_1 + y_2}{2} \\
\end{bmatrix} \approx \begin{bmatrix}
y_0 \\ y_1 \\ y_2 \\
\end{bmatrix} \quad \int_a^b f(x) \, dx \approx \frac{b-a}{6} \left( f(x_0) + 4f(x_1) + f(x_2) \right) + \frac{b-a}{2} \left( f(x_1) + f(x_2) \right) + \frac{b-a}{3} \left( f(x_2) + \cdots + f(x_{2n}) \right)
\]

integrate from \(-1\) to \(1\) (\( p(x) \) is our approximation for \( f(x) \))

\[
\frac{b-a}{6} \quad \text{special case of } \frac{b-a}{b}
\]

Surprise: models cubic functions perfectly (reason it is better than quadratic:

\[
\int_{-1}^{1} x^3 \, dx = 0.
\]

So error

\[
\int_a^b f(x) \quad \text{Simpson's rule for } f = \frac{(b-a)^5}{2880 \, n^4} \frac{f(4)}{c} \text{ for some } c \in (a, b).
\]

Also discussed Bernstein polynomials \( x^{n-k} (1-x)^k \) \( k = 0, \ldots, n \) instead of monomials.
Book also discusses Gaussian integration:

Again \( p \) of degree \( \leq d \). Pick points \( x_1, \ldots, x_m \) and weights \( w_1, \ldots, w_m \) so that, for all \( p \) of degree \( \leq d \) (idea: want \( m \) small)

\[
\int_{-1}^{1} p(x) \, dx = \sum_{i=1}^{m} w_i \, p(x_i) \quad (\text{Simpson: } x_i = -1, 0, 1 \quad w_i = 1, 4, 1)
\]

2m unknowns. Solve \( d+1 \) equations, one for \( 1, x, \ldots, x^d \).

Try to solve with \( 2m > d+1 \). If \( d = 3 \), can try with \( m = 2 \)

Get

\[
\int_{-1}^{1} 1 \, dx = w_1 + w_2 = 2
\]

\[
\int_{-1}^{1} x \, dx = w_1 x_1 + w_2 x_2 = 0
\]

\[
\int_{-1}^{1} x^2 \, dx = w_1 x_1^2 + w_2 x_2^2 = \frac{2}{3}
\]

\[
\int_{-1}^{1} x^3 \, dx = w_1 x_1^3 + w_2 x_2^3 = 0
\]

Bad because non-linear equations. Make some assumptions: \( x_1 = x \)

\( x_2 = -x \)

\( w_1 = w_2 = w \).

then eqns 2+4 are true. Left with

\[ 2w = 2, \quad 2wx^2 = \frac{2}{3} \quad \text{so} \quad w = 1 \quad x = \sqrt{\frac{1}{3}}. \]
Example:
\[
\int_{-1}^{1} \cos \frac{x}{\sqrt{2}} \, dx = \left. \sin \frac{2x}{\sqrt{2}} \right|_{-1}^{1} = \sin \frac{2\pi}{\sqrt{2}} - \sin \frac{\pi}{\sqrt{2}} = 2 \sin 1 \approx 1.6829.
\]

Simpson:
\[
= \frac{1}{3} \left( \cos \frac{2\pi}{\sqrt{2}} + 4 \cos \frac{0}{\sqrt{2}} + \cos \frac{2\pi}{\sqrt{2}} \right) = \frac{4}{3} + \frac{2}{3} \cos \frac{\pi}{\sqrt{2}} \approx 1.6935.
\]

Gaussian:
\[
= \frac{\cos \frac{\pi}{\sqrt{3}}}{\cos \frac{\pi}{\sqrt{3}}} + \frac{\cos \frac{-\pi}{\sqrt{3}}}{\cos \frac{\pi}{\sqrt{3}}} = 2 \cos \frac{\pi}{\sqrt{3}} \approx 1.6754.
\]

(also recall)
\[
\cos (x - \frac{\pi}{2}) = \sin x
\]
\[
\cos (\frac{\pi}{2} - x) = \sin x
\]
\[
\pi/2 - \sqrt{3} = 0.9934...
\]

Try to similarly handle higher degree by setting \( x_i = -x_{2m-1} + i \), eliminates all odd powered equation.

In practice, also use Gaussian rules for integrations like
\[
\int_{-\infty}^{\infty} f(x) e^{-x^2} \, dx,
\]
where we find answers for \( f \) of small degree as before.

Useful in probability. Only compute \( w_i, x_i \) once and from can use on all choices of \( f \).

In any case, all rules associate weights \( w_i \) and approximate values
\[
\sum_{i=1}^{k} w_i f(p_i) \quad p_i: \text{distinguished points in } [a,b].
\]

\[
\int_{a}^{b} f(x) \, dx \quad \text{Not Fubini.}
\]

Mon. Basic: 8.4.1

\[
\sum_{i} w_i f_1(p_i) \quad \sum_{i} w_i f_2(p_i)
\]

\[
\ln |R^2| := \int f(x) \, |dx| = \int f(x_1) \, dx_1 \int f_2(x_2) \, dx_2
\]

\[
|a,b|^2
\]

\[
|a,b|
\]
\[ \sum_{1 \leq i_1, i_2 \leq k} \frac{w_{i_1} w_{i_2}}{f_i(P_{i_1}) f_i(P_{i_2})} \]

So we can handle functions of several variables. Problem is that when dimension increases, sampling grows exponentially. Simpson, with 10 points a side for cube in dimension 8 gives 10^8 computations of special values of function. (Ok for fast computer. Any bigger we need another way.)

**Fix:** Sample points randomly in high dimensional cube and take average.

Know \[ \int_A f(x) \, d^nx = \frac{\text{average vol. of } f \text{ on } A}{\text{vol}(A)} \]

So sampling and averaging evaluates:

\[ \int_A f(x) \, d^nx \]

(if don't know volume of A in advance, put A in a box B and do Monte Carlo method for

\[ \int_B 1_A \, d^nx \, / \, \text{vol}(B) \].

Problem: How do we know if we're close to the right answer?

Ans: Approximate the standard deviation and use central limit theorem.