1. Let \( 1_D(x, y) \) denote the characteristic function on the unit disc \( D \) in \( \mathbb{R}^2 \). Is the function
\[
  f = \log_2(x^2 + y^2)1_D(x, y)
\]
Lebesgue integrable on \( \mathbb{R}^2 \)? Explain your answer.

**Yes.** Consider the partitioning of unit disc \( D \) by annuli with radii \( \frac{1}{2^k} \), \( \frac{1}{2^{k+1}} \), \( k = 0, 1, \ldots \)

so \( A_k = \{ (r, \theta) \mid r \in \left[ \frac{1}{2^k}, \frac{1}{2^{k+1}} \right] \} \).

Let \( f_k = f \cdot 1_{A_k} \). We claim that
\[
  \sum_{k=0}^{\infty} \int_{\mathbb{R}^2} |f_k| \, dx \, dy < \infty,
\]
so since \( \sum_{k} f_k = f \)

\[
  (*) \quad \text{then } f \text{ will be } \text{L-integrable}.
\]

Indeed \( \sup_{A_k} f_k = \log_2 \left( \frac{1}{2^k} \right) = -2k \)

so \( (*) \leq \sum_{k=0}^{\infty} 2k \cdot \text{vol}(A_k) \)

\[
  \leq \sum_{k=0}^{\infty} \frac{\pi \left( \frac{1}{2^k} \right)^2 - \pi \left( \frac{1}{2^{k+1}} \right)^2}{\pi \left( \frac{1}{2^k} \right)^2 - \pi \left( \frac{1}{2^{k+1}} \right)^2}
\]

which converges by the ratio test, e.g.
2. a) If $f$ is a Lebesgue integrable function on $\mathbb{R}^n$ and $g$ is a Riemann integrable function on $\mathbb{R}^n$, prove that the product $fg$ is Lebesgue integrable on $\mathbb{R}^n$.

Since $f$ is $L$-integrable, write it as $f = \sum_k f_k$, $f_k : R$-integrable

Thus $fg = \sum_k f_k \cdot g$. We must show $\sum_k \int_{\mathbb{R}^n} |f_k g| \, d^n x < \infty$.

But $g$ is bounded, since $R$-integrable, so $\sup |g| < \infty$. And

$$\sum_k \int_{\mathbb{R}^n} |f_k g| \, d^n x \leq \sup |g| \cdot \sum_k \int_{\mathbb{R}^n} |f_k| \, d^n x < \infty.$$ 

finite since $f$ is $L$-int.

b) Give sufficient conditions for which the function

$$F(t) = \int_{\mathbb{R}^n} f(t, x) \, |d^n x|$$

can be differentiated under the integral sign.

(Hint: To bring a limit inside an integral, use the dominated convergence theorem.)

$$\frac{d}{dt} F(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} = \lim_{h \to 0} \int_{\mathbb{R}^n} \frac{f(t+h, x) - f(t, x)}{h} \, |d^n x|$$

so to apply dominated convergence theorem, need for some $\epsilon > 0$

if $|h| < \epsilon$ then $\left| \frac{f(t+h, x) - f(t, x)}{h} \right| \leq g(x)$

for some $L$-integrable function $g$ for all $t$. Also need $\frac{d}{dt} f(t, x)$ defined a.e.
3. a) Exhibit a relaxed parametrization $\gamma$ for the surface of revolution $S$ obtained by rotating the curve
\[ z^2 = (1 + x)^3 \quad \text{with} \quad x \in [-1, 0] \]
about the $z$-axis in $\mathbb{R}^3$.

(Hint: First parametrize curve $z^2 = (1 + x)^3$ in $\mathbb{R}^2$ using a single parameter $t$. When you search for functions $f$ and $g$ so that $x = f(t)$ and $z = g(t)$, think about choosing simple polynomials so that the degrees of both sides of the curve equation match.)

To parametrize the curve, set $z = t^3$, $x = t^2 - 1$

then surface of revolution is $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$

\[
\begin{pmatrix}
  t^3 \\
  t^2 - 1 \\
  (t^2-1) \cos \theta \\
  (t^2-1) \sin \theta \\
  t^3
\end{pmatrix}
\]

We could do piecewise parametrization via $z = \pm \sqrt{(1+x)^2}$ with $x = t$, but this is much messier in the end.

b) Prove your answer in (a) gives a relaxed parametrization of $S$. In particular, find a set $X \subset U$ of points $x$ on which $\gamma : U \rightarrow S$ either:

- fails to be one-one, $C^1$, with Lipschitz derivative at $x$, or
- $[D\gamma(x)]$ is not one-one.

Explain why the set $X$ has volume 0.

These are smooth functions in parametrization $\gamma$ but they may fail to be one-one where curve touches $z$-axis in $x$-$z$ plane. Then all $\theta$ give same point. This occurs when $t = 1 - 1 = 0$ (places where $x$-coord = 0).

\[
[D\gamma(t, \theta)] = \begin{bmatrix}
  2t \cos \theta & -(t^2-1) \sin \theta \\
  2t \sin \theta & (t^2-1) \cos \theta \\
  3t^2 & 0
\end{bmatrix}
\]

which has non-trivial kernel when $t = 0$, $\theta$ arbitrary.

Finally, fails to be one-one when $\theta = 2\pi$ and $\theta = 0$
give same curves, again of vol. 0.

Circle of radius 1, which has volume 0.
c) Set up an iterated integral to compute the 2-volume of this surface of revolution.

(Your answer should have integrand in the form $\sqrt{p(t, \theta)}$ for some polynomial $p$ in $t$ and $\sin \theta$ and $\cos \theta$. But definitely don’t try to integrate this!)

$$
\begin{bmatrix}
2t \cos \theta & 2t \sin \theta & 3t^2 \\
-(t^2-1) \sin \theta & (t^2-1) \cos \theta & 0 \\
2t \cos \theta & -(t^2-1) \sin \theta & 0 \\
2t \sin \theta & (t^2-1) \cos \theta & 0 \\
3t^2 & 0 & 0
\end{bmatrix}
$$

\[
= \begin{bmatrix}
\frac{1}{4t^2} (\cos^2 \theta + \sin^2 \theta) + 9t^4 & 0 \\
0 & (t^2-1)^2 \cdot (\sin^2 \theta + \cos^2 \theta) \\
0 & 1
\end{bmatrix}
\]

\[
det = (4t^2 + 9t^4) \cdot (t^2-1)^2
\]

Compute

$$
\int_0^{2\pi} \int_{-1}^1 \sqrt{(4t^2+9t^4)(t^2-1)^2} \; dt \; d\theta.
$$
4. a) State Gauss’ Theorema Egregium on the area of discs $D_r(p)$ at a point $p$ on a 2-manifold $S$.

For $r$ sufficiently small

\[ D_r(p) = \pi r^2 - \frac{\pi}{12} K(p) r^4 + o(r^4) \]

b) Show that the theorem holds at the north pole $(x, y, z) = (0, 0, 1)$ of the unit sphere $x^2 + y^2 + z^2 = 1$ by computing both sides appearing in Gauss’ Theorem. You may use the fact that the disc $D_r(0, 0, 1)$ on the sphere projects to a disk of radius $\sin r$ in the $xy$-plane.

For computing $K(p)$ at north pole, we have

\[ z = f(x, y) = \sqrt{1 - x^2 - y^2} = 1 - \frac{1}{2} (x^2 + y^2) + o(x^2, y^2) \]

so $a_{2,0} = a_{0,2} = -1$ and its determinant is $K(p) = 1$.

To find surface area of $D_r(p)$:

Can either use spherical coords, or write surface as

\[ \begin{pmatrix} x \\ y \\ \sqrt{1 - x^2 - y^2} \end{pmatrix} = \begin{pmatrix} x \\ y \\ -\frac{1}{2} (x^2 + y^2) + o(x^2, y^2) \end{pmatrix} \]

then, using above fact, in $\rho, \theta$ coords in $x-y$ plane

\[ (\rho, \theta) \rightarrow (\rho \cos \theta, \rho \sin \theta) \quad \text{with} \quad U = \xi(\rho, \theta) \mid \rho < \sin \rho \]

\[ U \rightarrow \frac{\pi}{2} D_r(0, 0, 1) \]

So compute

\[ \int_0^{2\pi} \int_0^r \sqrt{\det(D\xi^T)(D\xi)} \, \rho \, d\rho \, d\theta = \pi r^2 + \frac{\pi}{12} r^4 + o(r^4) \]

requires simplifying using $\sqrt{1 + t} = 1 + \frac{1}{2} t + o(t)$. 