Math 8300 – Hecke Algebras – Problem Set 4  
Due: Monday, December 5

1. Let $G$ be the symplectic group $Sp_4$ over an algebraically closed field, as defined in Lecture 25.
   a) Determine the root datum associated to $G$ explicitly.
   b) Give pictures of the root subgroups $U_\alpha$ as matrices inside $GL_4$ for your choice of positive roots.
   c) Prove that the group $N_G(T)/T$ is isomorphic to the finite Coxeter group of type $B_2$ (two generators $s_1, s_2$ with $(s_1 s_2)^4 = 1$), where $T$ is the split torus contained in the diagonal matrices of $GL_4$.

2. In class, we had two definitions of induced representations for modules over the group algebra of a finite group (or equivalently modules for the group). One was via tensor products:
   \[ \text{Ind}_H^G(V) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \]
   and one via functions
   \[ \text{Ind}_H^G(V) = \{ f : G \to V \mid f(hg) = \pi(h) \cdot f(g) \} \]
   Show that the two definitions are equivalent.

3. Prove the geometric version of Mackey’s theorem given in Lecture 30:

   **Theorem 1 (Mackey)** Let $G$ be a finite group, with subgroups $H_1$ and $H_2$ having representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ respectively. Then $\text{Hom}_G(\text{Ind}_{H_1}^G(V_1), \text{Ind}_{H_2}^G(V_2))$ is isomorphic to the space of all functions $\Delta : G \to \text{Hom}_\mathbb{C}(V_1, V_2)$ satisfying:
   \[ \Delta(h_2gh_1) = \pi_2(h_2) \circ \Delta(g) \circ \pi_1(h_1). \]

4. Assuming Maschke’s theorem for Hecke algebras, as given in the notes for Lecture 31, give the details of the proof that the Hecke algebra $\mathcal{H}(G//B)$ over a field $F$ is semisimple, with $G = GL_n(\mathbb{F}_q)$ and $B$ the Borel subgroup of upper triangular matrices, if and only if
   \[ \text{char}(F) \nmid q \prod_{i=2}^n (q^i - 1) / (q - 1). \]
   In particular, compute the order of the group $G$ and the index of $B$ inside it, and give details for the proof sketched in Lecture 31 (again assuming Maschke’s theorem).